The critical probability for percolation on $\mathbb{L}^2$

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Contents

Chapter 1

Introduction

1.1 The aim of this report

This report is an end product of a semester project in mathematics at EPFL, Lausanne. No new theorems or other discoveries will be presented, all the material in this report can be found in books and articles regarding percolation theory. What this report aims to do is to give a complete explanation of the fundamental result that the critical probability in the square lattice is one half. A person who is aquainted with mathematics on an undergraduate/graduate level but has never heard of percolation before should be able to read this report. A few proofs are moved to an appendix, since they don't have anything to do with percolation theory.

1.2 What is percolation?

Every morning I put my coffee pot on the stove, turn up the heat and wait for the water to steam up through the grounded coffee beans. This movement or filtering of a fluid through a porous material is what a physicist or chemist calls percolation. The name for the mathematical model explored in this report originates from this process. In 1957 the first percolation model was formulated by Broadbent and Hammersley [1] for the question *"If a large porous stone is immersed in a bucket of water, does the centre of the stone get wet?"*.

For a start, imagine a lattice of channels through the rock. The lattice is made out of vertices that are connected by edges, in some manner. We then choose a parameter $p \in [0, 1]$, depending on what type of rock we have and what type of fluid we immerse it in, and declare the edges in the lattice open with probability p, closed otherwise. The liquid is allowed to flow through an open edge. If the scale of the lattice is tiny compared to the size of the rock, we can think of the lattice as infinite. Percolation theory treats questions about this situation. What is the probability that there is an open path from a point in the lattice, most often the origin, to infinity? What happens when we change p? What we will explore is that for small p there are alot of closed edges and there will (almost surely) not be an open path to infinity. But as we increase p a sharp threshold will occure and suddenly we have positive probability of having an infinite open path. The existance of this threshold makes percolation a rich subject, and makes it possible for percolation to serve as a model for systems that undergo a phase transition. In all cases of phase transition, there is a unit that becomes non-zero above (or below) a critical point. For percolation the unit is the probability that there exists an infinite cluster and the critical point called the critical probability. One example of a physical system where phase transition occurs is a magnetic system, where the unit is the spontaneous magnetisation per spin, and the critical point is the Curie temperature T_c. When $T < T_c$ the spontaneous magnetisation per spin becomes non-zero and the magnetic system enters an ordered state. Another example is the Van der Waal equation for fluids. The unit here is the difference between liquid and vapour density, which becomes non-zero as the temperature goes below some critical temperature. In both examples above and the percolation model we will study in this report, the unit goes to zero continuously as it approaches the critical point. Such a phase transition is called a second-order phase transition. There are systems where the unit jumps as it approaches the critical point. An example is the density of various materials when they undergo solid/liquid/gas transitions. These phase transitions are called first-order phase transitions and can not be modeled by our percolation model, but requires more elaborate percolation models.

1.3 Basic concepts

Percolation can be modeled on any kind of graph (honeycomb, tree, Bethe, ect.), but in this report we will only study the cubic lattice. The cubic lattice is constructed by choosing $\mathbb{Z}^d = \{v = (v_1, v_2, \ldots, v_d) : v_i \in \mathbb{Z}, i = 1, 2, \ldots, d\}$ to be the vertex set. We define the distance between two vertices in $\mathbb{Z}^d$ to be

$$\delta(v, v') = \sum_{i=1}^{d} |v_i - v'_i|. \tag{1.1}$$

By adding edges between all pairs v, v' with $\delta(v, v') = 1$ we turn $\mathbb{Z}^d$ into a graph and δ to the standard graph theoretic distance. The edge set will be denoted E and the cubic lattice will be denoted $\mathbb{L}^d = (\mathbb{Z}^d, E)$.

Let $p \in [0, 1]$. Each edge in E is declared open with probability p, otherwise closed, and all edges are taken to be independent of each other. With the porous rock example in mind, an open edge allows passage for the water from one hole (vertex $v \in \mathbb{Z}^d$) to another but a closed edge does not. This will give us a probability measure on the set of subsets of $\mathbb{L}^d$ if we have a sample space and a σ-algebra. Let an outcome in the sample space be the bond configuration 0/1-vector ω

$$\begin{aligned} \omega : E &\rightarrow (\omega(e_1), \omega(e_2), \dots) \\ e &\mapsto \omega(e) \end{aligned}$$

A bond $e \in E$ is open in the configuration ω if and only if $w(e) = 1$ and closed otherwise. The sample space is the set $\Omega = \{0, 1\}^E$ of all configurations ω. Let Σ be the σ-field generated on Ω by finite dimensional cylinder sets

$$C(F, \gamma) = \{\omega \in \Omega : \omega(f) = \gamma(f),\ f \in F\}$$

where F is a finite subset of E^d and $\gamma \in \{0, 1\}^F$. Why this is the choice of σ-algebra is motivated in the Appendix.

With p as above in mind, the relevant probability measure on (Ω, Σ) is the product measure $\mathbb{P}_p$ induced by

$$\mathbb{P}_p(C(F, \gamma)) = \prod_{\substack{f \in F \\ \gamma(f)=1}} p \prod_{\substack{f \in F \\ \gamma(f)=0}} (1 - p).$$

This is the underlying structure we will work on throughout this report. From it we can define some important concepts in percolation theory.

For two vertices $x, y \in \mathbb{Z}^d$, we write $x \leftrightarrow y$ if there exists an open path joining x and y. Note that $x \leftrightarrow y$ is an event in Ω and can therefore be measured.

Definition 1.3.1 *The open cluster C_x at x is the set of all vertices reachable along open paths from the vertex x,*

$$C_x = \{y \in \mathbb{Z}^d : x \leftrightarrow y\}$$

The open cluster from the origin is abbreviated to C. Often we want to know what the probability is that C is infinite. For this we define the percolation probability.

Definition 1.3.2 *The percolation probability $\theta(p)$ is the probability that there will exists an infinite open cluster containing the origin,*

$$\theta(p) = \mathbb{P}_p(|C| = \infty)$$

Later it will be shown that the event $\mathbb{P}_p|C_x| = \infty$ is translation invariant in the lattice, and hence the percolation probability is independent of the at what vertex x we are looking.

The most interesting concern about the percolation probability is at what p it is positive. Later on we will show that $\theta(p)$ is non decreasing in p. This motivates the definition of the critical probability which is the subject of the main theorem in this report.

Definition 1.3.3 *The critical probability, p_c, is*

$$p_c = \sup\{p : \theta(p) = 0\}$$

1.4 Examples

After the foundations in the previous section, it is time to get comfortable with the percolation probability and the critical probability. The fundamental problem in percolation theory is to find p_c for a certain lattice, that is to ask the question *"when does percolation occur?"*. It is not a trivial task to do this for a general lattice, and it is not even proved that $\theta(p_c) = 0$ for all cubic lattices [2]. It makes sense to ask this question though, and we will se why in the following thorem.

Theorem 1.4.1 *The probability of an infinite open cluster on $\mathbb{L}^2$ is either* 0 *or* 1.

Proof This theorem is proved by using Kologorov's zero-one law. In order to make use of Kolmogorov's zero-one law we need to create a sequence of σ-algebras on Ω generated by independent events such that the existance of an infinite open cluster is in the tail of this sequence. Since our sample space is countable we can enumerate all edges in some way. Now let A_i be the event that the ith edge is open and

$$\Sigma'_n = \sigma(A_1, A_2, \ldots, A_n).$$

Our sought family is $(\Sigma'_n)_n$.

A finite number of closed edges can only disconnect a finite number of vertices from the rest of $\mathbb{Z}^2$. Therefore the existance (or non-existance) of an infinite open cluster can not be decided by a finite subset of all A_i's. Therefore the event that there is an infinite open cluster is in the tail of Σ'. By Kolmogorov's zero-one law it has then either probability 0 or probability 1.

□

The next example is about percolation on $\mathbb{L}$, and the result is very intuitive.

Proposition 1.4.2 *When $d = 1$, $p_c(1) = 1$.*

Proof. Enumerate the vertices by their position in the lattice, i.e. the node to the right of the origin is called 1, the next one 2, ect. For nonnegative integers i, let A_i be the event that $2^i \leftrightarrow 2^{i+1} - 1$ and $-2^{i+1} + 1 \leftrightarrow -2^i$ happens. The probability for this event is

$$\mathbb{P}_p(A_i) = p^{2^i} \cdot p^{2^i}.$$

When $p < 1$ the sum over all A_i's is convergent,

$$\sum_{i=1}^{\infty} \mathbb{P}_p(A_i) < \infty.$$

The Borel-Cantelli lemma then tells us that $\mathbb{P}_p(A_i \text{i.o.}) = 0$, in other words that there must exist an i for which A_i does not happen with probability 1. Thus the probability of an infinite component when $p < 1$ is zero.

□

Chapter 2

Increasing events, inequalities and influence

In this chapter we first identify a class of events that will occur in percolation again and again as we go deeper into the theory, increasing events. Together with these follow increasing random variables. For this class of events and random variables we will prove some useful inequalities that in the end of this chapter give rise to a theorem about the change of the number of infinite open clusers when we are changing p. We will see that there is a sharp transition occurs, and this is the threshold that was mentioned in the introduction.

2.1 Increasing events

Before we state the definition of an increasing event, let us develop an intuition for what they are. Let the samples in our sample space have a partial order in the following way. For two configurations $\omega_1, \omega_2 \in \Omega$,

$$\omega_1 \leq \omega_2 \quad \text{if} \quad \omega_1(e) \leq \omega_2(e) \; \forall e \in E.$$

In other words, if an edge is open in ω_1 it must be open in ω_2. Consider now the event $A \subset \Omega$ that the origin is contained in an infinite open cluster. If A happens in a configuration ω, it will still happen if we open up edges that are closed in ω. With the inequality defined above, this can be expressed as if there is an $\omega_1 \in A$ and an $\omega_2 \in \Omega$ such that $\omega_1 \leq \omega_2$ then the set of open edges in ω_2 is a superset of the open edges in ω_1, and thus $\omega_2 \in A$. This event is an example on an increasing event.

Definition 2.1.1 *A non-empty subset $A \subset \Omega$ is called an increasing event if*

$$\omega_1 \in A, \omega_1 \leq \omega_2 \Rightarrow \omega_2 \in A.$$

A decreasing event is defined in the same way but with the inequality in the other direction. Lets look at examples of increasing events.

Example 1. An important example of an increasing event is $x \leftrightarrow y$, i.e. the event that there exists an open path from a vertex x to a vertex y. If the open path exist in a configuration ω and we open up edges in ω, there open path will still exist.

Example 2. The event $A = \{\omega \in \Omega : |\text{open edges in } \omega| \geq k\}$ is an increasing event. If there are k open edges in ω, then if $\omega' \geq \omega$, there must be more or equally many open edges in ω'. Hence ω' is also in A and the event is increasing. Note that if the $\geq$ condition in A is changed to for example $=$, $\leq$ or *"is even"*, A will no longer an increasing event.

If Ω is finite the $\geq$ condition can be exchanged to *"is finite"* and A would still be an increasing event, but not if Ω is infinite.

Example 3. An intersection of increasing events is an increasing event. If A and B are increasing events and $\omega \in A \cap B$, then $\omega \in A$ and $\omega \in B$. Since A and B are increasing, all ω' such that $\omega' \geq \omega$ will be both in A and B and hence in $A \cap B$.

Also a union of increasing events in an increasing event. If A is an increasing event, we can replace A by any superset of A in the definiton of increasing events and the definition would still hold. Our claim follows since the union is a superset of its parts.

Using this definition and a technique called coupling, we will now prove a result about the percolation probability.

Proposition 2.1.1 *$\theta(p)$ is a non-decreasing function in p.*

Proof. We will prove a result for increasing events and then apply it to $|C| = \infty$. For each $e \in E$, define a random variable $X(e)$ with a uniform distribution in $[0, 1]$. Define

$$\alpha_p(e) = \begin{cases} 0, & X(e) > p \\ 1, & X(e) \leq p \end{cases}$$

The state of each edge will be assigned from the value of α_p. Generate two configurations like this, with probabilities p_1 and p_2. Define

$$\begin{aligned}\omega_{p_1} &= \alpha_{p_1}(e_1)\alpha_{p_1}(e_2)\ldots,\\ \omega_{p_2} &= \alpha_{p_2}(e_2)\alpha_{p_2}(e_2)\ldots.\end{aligned}$$

These events can be either 0 or 1, and by definition $\omega_{p_1} \leq \omega_{p_2}$ whenever $p_1 \leq p_2$. Therefore, for any increasing event A we have by definition $\omega_{p_1} \in A \Rightarrow \omega_{p_2} \in A$. Our probability measure is the product measure, and hence we this general result for increasing events

$$P_{p_1}(A) \leq P_{p_2}(A)$$

Since $\theta(p) = \mathbb{P}_p(|C| = \infty)$ and $|C| = \infty$ is an increasing event the proposition is proved.

□

2.2 FKG inequality

The FKG inequality states that two increasing events have positive correlation. From the previous chapter, we know that the event $x \leftrightarrow y$ is an increasing event. It is reasonable that if $x \leftrightarrow y$ happens, then it becomes more likely for $z \leftrightarrow w$ to happen. This was proved for product measures by Harris [3]. A generalization for non-product measures was done by Fortuin, Kasteleyn and Ginibre [4] and this is where the inequality got its name from. We will use this inequality in the proof of the main theorem, and since we work with product measures there we stick to Harris' version of the inequality.

Theorem 2.2.1 *If A and B are increasing events then*

$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B). \tag{2.1}$$

If X and Y are increasing random variables then

$$\mathbb{E}_p[XY] \geq \mathbb{E}_p[X]\,\mathbb{E}_p[Y]. \tag{2.2}$$

Proof. It is sufficient to prove (2.2) because (2.1) then follows if we let the increasing random variables be indicator functions of increasing events. In the first part of the proof it will be shown by induction on n, the number of edges that X and Y depend on, that (2.2) holds for all finite n. In the second part we take the step to countably infinite n by using a martingale convergence theorem.

Part 1. Suppose X and Y depend only on $\omega(e_1), \omega(e_2), \ldots, \omega(e_n)$ and suppose that $n = 1$. In this situation, that we will use as a basis for the induction, X and Y do only depend on one edge e_1 which has two states, 0 and 1. Let ω_1 and ω_2 be two different configurations of Ω that differ on e_1. Since X and Y are increasing random variables the signs of $X(\omega_1(e_1)) - X(\omega_2(e_1))$ and $Y(\omega_1(e_1)) - Y(\omega_2(e_1))$ will be the same. Therefore

$$[X(\omega_1(e_1)) - X(\omega_2(e_1))]\,[Y(\omega_1(e_1)) - Y(\omega_2(e_1))] \geq 0.$$

Taking the sum over all values of $\omega_1(e_1)$ and $\omega_2(e_1)$, and multiplication with their probability measure (which is positive) gives

$$\begin{aligned} 0 &\leq \sum_{\omega_1(e_1)=0}^{1} \sum_{\omega_2(e_1)=0}^{1} [X(\omega_1(e_1)) - X(\omega_2(e_1))]\,[Y(\omega_1(e_1)) - Y(\omega_2(e_1))] \\ &\quad \times \mathbb{P}_p(\omega(e) = \omega_1(e_1))\mathbb{P}_p(\omega(e) = \omega_2(e_1)) \\ &= \ldots \\ &= 2\left(\mathbb{E}_p[XY] - \mathbb{E}_p[X]\,\mathbb{E}_p[Y]\right), \end{aligned}$$

which proves the theorem when $n = 1$. Now assume that (2.2) holds for all $n < k$, and assume that X and Y are increasing in $\omega(e_1), \omega(e_2), \ldots, \omega(e_{k-1})$. Then from the tower property of conditional expectation and the base case, we get

$$\begin{aligned} \mathbb{E}_p[XY] &= \mathbb{E}_p\left[\mathbb{E}_p[XY \mid \omega(e_1), \ldots, \omega(e_{k-1})]\right] \\ &\geq \mathbb{E}_p\left[\mathbb{E}_p[X \mid \omega(e_1), \ldots, \omega(e_{k-1})]\,\mathbb{E}_p[Y \mid \omega(e_1), \ldots, \omega(e_{k-1})]\right] \\ &\geq \mathbb{E}_p\left[\mathbb{E}_p[X \mid \omega(e_1), \ldots, \omega(e_{k-1})]\right]\mathbb{E}_p\left[\mathbb{E}_p[Y \mid \omega(e_1), \ldots, \omega(e_{k-1})]\right] \\ &= \mathbb{E}_p[X]\,\mathbb{E}_p[Y]. \end{aligned}$$

Part 2. Assume now that X and Y are increasing random variables with finite second moments. Define

$$X_n = \mathbb{E}_p[X \mid \omega(e_1), \ldots, \omega(e_n)], \quad Y_n = \mathbb{E}_p[Y \mid \omega(e_1), \ldots, \omega(e_n)]. \tag{2.3}$$

Note that X_n and Y_n in (2.3) are martingales w.r.t. the σ-algebra that they generate. As shown in Part 1, X_n and Y_n are increasing in $\omega(e_1), \ldots, \omega(e_n)$ and therefore

$$\mathbb{E}_p[X_n Y_n] \geq \mathbb{E}_p[X_n]\,\mathbb{E}_p[Y_n]. \tag{2.4}$$

This is the relation we would like to be preserved when we take the limit $n \to \infty$. We need a well known theorem about convergence of martingales to show this. A proof can be found in [5].

Theorem 2.2.2 *If X_n is a martingale with finite second moment for all n, then there exists a random variable X such that X_n converges to X almost surely and in mean square.*

A straight away consequence of Theorem 2.2.2 is that nothing happens with the left hand side of (2.4) as we go in the limit,

$$\mathbb{E}_p[X_n] \to \mathbb{E}_p[X], \quad \mathbb{E}_p[Y_n] \to \mathbb{E}_p[Y] \quad \text{as } n \to \infty.$$

The right hand side requires us to use the triangle inequality and Cauchy-Schwartz inequality.

$$\begin{aligned} \mathbb{E}_p[|X_nY_n - XY|] &\leq \mathbb{E}_p[|(X_n - X)Y_n| + |X(Y_n - Y)|] \\ &\leq \sqrt{\mathbb{E}_p[(X_n - X)^2]\,\mathbb{E}_p\left[Y_n^2\right]} + \sqrt{\mathbb{E}_p[X^2]\,\mathbb{E}_p[(Y_n - Y)^2]} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence, if we take the limit of (2.4) we obtain (2.2).

□

We conclude this section with an application of the FKG inequality. It is a proposition about the infinite open cluster.

Proposition 2.2.3 *The probability that a vertex lies in an infinite open cluster is translational invariant. That is, for any two vertices $x, y \in \mathbb{L}^d$,*

$$p_c(x) = p_c(y). \tag{2.5}$$

Proof. If we know that x lies in an infinite open cluster, we can not say anything about whether y lies in an infinite open cluster or not. But we have the relation

$$(x \leftrightarrow y) \cap (y \leftrightarrow \infty) \subset (x \leftrightarrow \infty).$$

Translating into probabilities, we get

$$\mathbb{P}_p(\,(x \leftrightarrow y) \cap (y \leftrightarrow \infty)\,) \leq \mathbb{P}_p(x \leftrightarrow \infty).$$

But as shown in section 2.1, $x \leftrightarrow y$ and $y \leftrightarrow \infty$ are increasing events. Therfore we can apply the FKG inequality to the last equation,

$$\mathbb{P}_p(x \leftrightarrow y)\mathbb{P}_p(y \leftrightarrow \infty) \leq \mathbb{P}_p(x \leftrightarrow \infty).$$

Since $\mathbb{P}_p(x \leftrightarrow y) > 0$, the probability that y lies in an infinite open cluster is 0 whenever the probability that x lies in an infinite open cluster is 0. In critical probabilities, this translates to $p_c(y) \geq p_c(x)$. If we swap x with y and do the whole argument again, we get the opposite inequality $p_c(x) \geq p_c(y)$. The two inequalities prove the proposition.

□

2.3 Influence

We can imagine the situation that the state of one or more edges can decide whether an event happens or not. An example is the event $0 \leftrightarrow \partial B(n)$ where $\partial B(n)$ is the boundary of the $2n \times 2n$-square centered at the origin. Lets say we generate a percolation in the same manner as in Proposition 2.1.1, and $0 \leftrightarrow \partial B(n)$ is close to happening. By this we mean, if we flip the state of a couple of edges we can make the event happen. These "influential" edges are what we will study in this section, and they will lead us to theorems about sharp thresholds.

Grimmett [2] defines the influence of an element $e \in E$ on the outcome of an event A in two ways, conditional- and absolute influence. These two are equivalent for increasing events under a product measure. This is always the situation in this report, and therefore we will not distinguish between the definitions.

Definition 2.3.1 *The influence of an element $e \in E$ on an event A is*

$$I_A(e) = \mathbb{P}_p(1_A(\omega^e) \neq 1_A(\omega_e)) \tag{2.6}$$

where

$$\omega^e = \begin{cases} \omega(f) & \text{if } f \neq e, \\ 1 & \text{if } f = e, \end{cases} \qquad \omega_e = \begin{cases} \omega(f) & \text{if } f \neq e, \\ 0 & \text{if } f = e. \end{cases}$$

In words, ω^e is the configuration ω with the edge e forced to be open, and ω_e with e forced to be closed. For increasing events (2.6) can be rewritten into a more convenient form.

Proposition 2.3.1 *For an increasing event A*

$$I_A(e) = \mathbb{P}_p(A^e) - \mathbb{P}_p(A_e), \tag{2.7}$$

where

$$A^e = \{\omega \in \Omega : \omega^e \in A\}, \quad A_e = \{\omega \in \Omega : \omega_e \in A\}.$$

Proof. By definition, $\omega^e \geq \omega_e$ implies that $\omega^e \in A$ whenever $\omega_e \in A$ and thus $\{1_A(\omega^e) \neq 1_A(\omega_e)\} = \{\omega \in \Omega : \omega^e \in A,\ \omega_e \notin A\}$. Using this together with De

Morgan's laws and the additivity of the probability measure we rewrite (2.6).

$$
\begin{aligned}
I_A(e) &= \mathbb{P}_p\left(1_A(\omega^e) \neq 1_A(\omega_e)\right) \\
&= \mathbb{P}_p\left(\{\omega \in \Omega : \omega^e \in A, \omega_e \notin A\}\right) \\
&= \mathbb{P}_p\left(\{\omega \in \Omega : \omega^e \in A\} \cap \{\omega \in \Omega : \omega_e \notin A\}\right) \\
&= \mathbb{P}_p\left(\{\omega \in \Omega : \omega^e \in A\}^C \cup \{\omega \in \Omega : \omega_e \notin A\}^C\right)^C \\
&= 1 - \mathbb{P}_p\left(\{\omega \in \Omega : \omega^e \in A\}^C \cup \{\omega \in \Omega : \omega_e \in A\}\right) \\
&= 1 - \left[1 - \mathbb{P}_p\left(\{\omega \in \Omega : \omega^e \in A\}\right) + \mathbb{P}_p\left(\{\omega \in \Omega : \omega_e \in A\}\right)\right] \\
&= \mathbb{P}_p(A^e) - \mathbb{P}_p(A_e).
\end{aligned}
$$

□

A very non trivial theorem has been proven about the influence. We can write down lower bounds for the total influence of all edges and the maximum influence of one edge. The proof is long and uses techniques from other fields of mathematics.

Theorem 2.3.2 *There exists a constant $c \in (0, \infty)$ such that if $|E| = N \geq 1$ and if $A \subset \Omega$ with probability $\mathbb{P}_p(A) \in (0, 1)$, then*

$$\sum_{e \in E} I_A(e) \geq c\mathbb{P}_p(A)(1 - \mathbb{P}_p(A)) \log\left(\frac{1}{\max_e I_A(e)}\right). \tag{2.8}$$

Furthermore there exists an edge $e \in E$ such that

$$I_A(e) \geq c\mathbb{P}_p(A)(1 - \mathbb{P}_p(A))\frac{\log(N)}{N}. \tag{2.9}$$

Proof. The proof will be divided into three parts. In the first part, we apply discrete fourier analysis to pave the way for the second part of the proof. There we use a result called the Hypercontractivity Lemma, often attributed to Bonami, Gross or Beckner, to bound the influence from below. In the third part we show the second assertion of the theorem.

Part 1. Define for functions from the sample space to the reals

$$< f, g >= \mathbb{E}_p\left[fg\right], \quad f, g : \Omega \to \mathbb{R}. \tag{2.10}$$

We are interested in a subset of these functions, namely the ones who take values in $\{0, 1\}$. These functions are called Boolean functions. The Boolean functions on Ω are in one to one correspondence to the power set of E via the relation $f = 1_A \leftrightarrow A \in \mathcal{P}(E)$. Grimmett [2] calls (2.10) for an inner product, but it is not an inner product since the space of Boolean functions is not a vector space. Let us anyway show that (2.10) satisfies the definition of an inner product, with the exception that is does not take elements from a vector space. Let $f = 1_A, g = 1_B, h = 1_C, A, B, C \in \Omega$ and $k \in \mathbb{R}$

- $\mathbb{E}_p\left[f^2\right] = \mathbb{E}_p\left[1_A 1_A\right] = \mathbb{P}_p(A) \geq 0$ and only $= 0$ when $A = \emptyset$,
- $\mathbb{E}_p\left[fg\right] = \mathbb{E}_p\left[1_A 1_B\right] = \mathbb{E}_p\left[1_B 1_A\right] = \mathbb{E}_p\left[gf\right]$,
- $\mathbb{E}_p\left[k(f+h)g\right] = k\mathbb{E}_p\left[fg + hg\right] = k\mathbb{E}_p\left[1_A 1_B + 1_A 1_C\right] = k\left(\mathbb{E}_p\left[1_A 1_B\right] + \mathbb{E}_p\left[1_A 1_C\right]\right) = k\left(\mathbb{E}_p\left[fg\right] + \mathbb{E}_p\left[hg\right]\right)$.

Hence (2.13) induces a L^2-norm,

$$||f||_2 = \sqrt{< f, f >} = \sqrt{\mathbb{E}_p\left[f^2\right]}. \tag{2.11}$$

We will use the L^2-norm later to write down Parseval's relation.

The next step towards a working fourier analysis is to choose a set of basis functions on the subsets of E. For $F \subset E$, let

$$u_F(\omega) = \prod_{e \in F} (-1)^{\omega(e)} = (-1)^{\sum_{e \in F} \omega(e)}, \quad \omega \in \Omega \tag{2.12}$$

Proposition 2.3.3 *$\{u_F\}_{F \subset E}$ is an orthonormal basis for Boolean functions on Ω.*

Proof. Identify F as a subset of the natural number in the following manner. Enumerate the edges in E. We can now identify a subset $F \subset E$ with a N-vector where a one indicates that the corresponding edge is in F, and zero otherwise. For example, $F = (1, 0, 1, 0, \ldots, 0)$ is a subset of two edges. For our purpose, we denote $F = \{1, 3\}$. Then for a configuration ω, we have that

$$u_F(\omega) = (-1)^{F \cdot \omega} = (-1)^{\sum_{i=1}^{n} F_i \omega_i} = (-1)^{\sum_{i \in F} \omega_i},$$

where ω_i is the ith coordinate of ω, which is either 0 or 1. The "inner product" between two us is thus

$$< u_F(\omega), u_G(\omega) >= \mathbb{E}_p\left[(-1)^{\sum_{i \in F} \omega_i} (-1)^{\sum_{i \in G} \omega_i}\right] = \mathbb{E}_p\left[(-1)^{\sum_{i \in F} \omega_i + \sum_{i \in G} \omega_i}\right]. \tag{2.13}$$

Now we are faced with two cases. If $F = G$ then the exponent in the last term of (2.13) becomes a square and the "inner product" will be equal to 1. For the other case, $F \neq G$, note that the probability of the exponent in the last term of (2.13) to be even is the same as the probability of it to be odd. This can be seen for example by looking at the expectation of $\sum_{i\in F}\omega_i$. We get

$$\begin{aligned}\mathbb{E}_p\left[(-1)^{\sum_{i\in F}\omega_i+\sum_{i\in G}\omega_i}\right] &= 1\cdot\mathbb{P}_p(\sum_{i\in F}\omega_i+\sum_{i\in G}\omega_i \text{ is even}) \\ &\quad +(-1)\cdot\mathbb{P}_p(\sum_{i\in F}\omega_i+\sum_{i\in G}\omega_i \text{ is odd}) = 0.\end{aligned}$$

□

Since $\{u_F\}_{F\subset E}$ is an orthonormal basis, every Boolean function $f:\Omega\to\{0,1\}$ can be expressed as a sum $f=\sum_{F\subset E}\hat{f}(F)u_F$ where $\hat{f}(F) =< f,u_F >$. In particular

$$\hat{f}(\emptyset) =< f,u_\emptyset >= \mathbb{E}_p\left[fu_\emptyset\right] = \mathbb{E}_p\left[f(-1)^0\right] = \mathbb{E}_p\left[f\right]$$

and

$$\begin{aligned}< f,g > &= \mathbb{E}_p\left[fg\right] = \mathbb{E}_p\left[\sum_{F\subset E}\hat{f}(F)u_F\sum_{G\subset E}\hat{g}(G)u_G\right] \\ &= \mathbb{E}_p\left[\sum_{F\subset E}\hat{f}(F)\hat{g}(F)u_F^2\right] \\ &= \sum_{F\subset E}\hat{f}(F)\hat{g}(F)\mathbb{E}_p\left[u_F^2\right] \\ &= \sum_{F\subset E}\hat{f}(F)\hat{g}(F). \end{aligned}\tag{2.14}$$

We used the linearity of the expectation and the fact that u_F^2 is deterministic and equal to one. By letting $f=g$ in (2.14) we get Parseval's relation

$$\|f\|_2^2 =< f,f >= \sum_{F\subset E}\hat{f}(F)^2. \tag{2.15}$$

Towards a formula for the total influence, we need to introduce one more definition. Let $f_e(\omega) = f(\omega)-f(\kappa_e\omega)$ where $\kappa_e\omega$ is the configuration ω with the state of e flipped. Since the image of f_e is $\{-1,0,1\}$ it holds that $|f_e| = f_e^2$. With $B=\{e\in E : \omega(e)=1\}$, the Fourier coefficients to f_e are given by

$$\begin{aligned}\hat{f}_e(F) &= < f_e,u_F >= \frac{1}{2^N}\sum_{\omega\in\Omega}f_e(\omega)u_F(\omega) = \frac{1}{2^N}\sum_{\omega\in\Omega}\left(f(\omega)-f(\kappa_e\omega)\right)u_F(\omega) \\ &= \frac{1}{2^N}\sum_{\omega\in\Omega}\left(f(\omega)-f(\kappa_e\omega)\right)(-1)^{|B\cap F|} \\ &= \frac{1}{2^N}\sum_{\omega\in\Omega}f(\omega)\left((-1)^{|B\cap F|}-(-1)^{|(B\Delta\{e\})\cap F|}\right).\end{aligned}$$

The last rewriting may seem taken out of the blue, but it has a nice structure. We have to notice three things.

- If $e \notin F$ then $|B \cap F| = |(B\Delta\{e\}) \cap F|$,
- if $e \in F$ and $e \in B$ then $|(B\Delta\{e\}) \cap F| = |(B - \{e\}) \cap F| = |B \cap F| - 1$,
- and lastly if $e \in F$ but $e \notin B$ then $|(B\Delta\{e\}) \cap F| = |(B \cup \{e\}) \cap F| = |B \cap F| + 1$.

This gives us

$$(-1)^{|B\cap F|} - (-1)^{|(B\Delta\{e\})\cap F|} = \begin{cases} 0, & e \notin F \\ 2u_F, & e \in F \end{cases}$$

so that

$$\hat{f}_e(F) = \begin{cases} 0, & e \notin F \\ 2\hat{f}(F), & e \in F \end{cases}$$

Now since $I_A(e) = \mathbb{P}_p(A^e) - \mathbb{P}_p(A_e) = \mathbb{E}_p[f_e] = \|f_e\|_2^2$ we get

$$I_A(e) = \|f_e\|_2^2 = 4 \sum_{F:e\in F} \hat{f}(F)^2$$

and the total influence is thus

$$\sum_{e\in E} I_A(e) = 4 \sum_{F\subset E} |F|\hat{f}(F)^2. \tag{2.16}$$

Part 2. Time to bound the right hand part of (2.16) from below. The idea is to find an inequality for large subsets, which will be easy, and to find another inequality for small subsets, which will require some work. Let b be a constant, to be determined later, that will be the size where we split small subsets from large ones. From (2.16) we get a bound for large subsets

$$\sum_{e\in E} I_A(e) \geq 4b \sum_{F\subset E:|F|>b} \hat{f}(F)^2 \tag{2.17}$$

This was the easy inequality. The main tool in finding an inequality for small sets will be the so called Hypercontractivity Lemma. To state it, we need to define a general norm for our function space and the so called noise operator. For $w \in [1, \infty)$, the L^w-norm is defined as

$$\|g\|_w = \mathbb{E}_p\left[|g|^w\right]^{1/w}, g : \Omega \to \mathbb{R}.$$

The main ingredient in the Hypercontractivity Lemma is the noise operator T_ρ. It is called the noise operator because it smoothens functions that are otherwise noisy. It is defined as follows. For $\rho \in \mathbb{R}$, let

$$T_\rho g = \sum_{F \subset E} \hat{g}(F)\rho^{|F|} u_F. \tag{2.18}$$

The norm of the noise operator satisfies

$$\|T_\rho g\|_2^2 = \sum_{F \subset E} \hat{g}(F)^2 \rho^{2|F|}. \tag{2.19}$$

We are ready to state the Hypercontractivity lemma. The proof is presented in the Appendix.

Lemma 2.3.4 *For $g : \Omega \to 0, 1$ and $\rho > 0$*

$$\|T_\rho g\|_2 \leq \|g\|_{1+\rho^2}. \tag{2.20}$$

Now let $0 < \rho < 1$ and $g = f_e$, where as before $f = 1_A$. Then

$$\begin{aligned} \sum_{\substack{F \subset E \\ e \in F}} 4\hat{f}(F)^2 \rho^{2|F|} &= \sum_{F \subset E} \hat{f}_e(F)^2 \rho^{2|F|} = \|T_\rho f_e\|_2^2 \leq \|f_e\|_{1+\rho^2}^2 \\ &= \mathbb{E}_p \left[|f_e|^{1+\rho^2} \right]^{2/(1+\rho^2)} = \|f_e\|_2^{4/(1+\rho^2)} = I_A(e)^{2/(1+\rho^2)}. \end{aligned}$$

Let $t = \mathbb{P}_p(A) = \hat{f}(\emptyset)$ and b as in the inequality for large subsets. Then

$$\sum_{e \in E} I_A(e)^{2/(1+\rho^2)} \geq 4\rho^{2b} \sum_{\substack{F \subset E \\ 0 < |F| \leq b}} \hat{f}(F)^2 = 4\rho^{2b} \left(\sum_{\substack{F \subset E \\ |F| \leq b}} \hat{f}(F)^2 - t^2 \right). \tag{2.21}$$

If we add together the inequalities (2.17) and (2.21) we get the total inequality

$$\rho^{-2b} \sum_{e \in E} I_A(e)^{2/(1+\rho^2)} + \frac{1}{b} \sum_{e \in E} I_A(e) \geq 4 \sum_{F \subset E} \hat{f}(F)^2 - 4t^2 = 4t(1-t). \tag{2.22}$$

In the last equality we used Parseval's relation (2.15). The final step is to choose $\rho = 1/2$. Then, under the assumption that $\max_e I_A(e) < 1$

$$\sum_{e \in E} I_A(e)^{4/3} \leq \left(\max_e I_A(e) \right)^{1/3} \sum_{e \in E} I_A(e)$$

and

$$\left(2^b (\max_e I_A(e))^{1/3} + b^{-1} \right) \sum_{e \in E} I_A(e) \geq 4t(1-t). \tag{2.23}$$

If we pick b such that $2^b\,(\max_e I_A(e))^{1/3} = b^{-1}$, we get

$$\begin{aligned} & 2^{b+\log(b)}\left(\max_e I_A(e)\right)^{1/3} = 1 \\ \Rightarrow\quad & b + \log(b) = \frac{1}{3}\log\left(1/\max_e I_A(e)\right) \\ \Rightarrow\quad & b \geq A\log\left(1/\max_e I_A(e)\right), \quad A > 0. \end{aligned}$$

If we insert this value into (2.23) we arrive to the sought inequality with $c = 2A$,

$$\sum_{e\in E} I_A(e) \geq \frac{4t(1-t)}{2b^{-1}} \geq 2A\mathbb{P}_p(A)(1-\mathbb{P}_p(A))\log\left(1/\max_e I_A(e)\right).$$

Part 3. The second assertion of the theorem is easily shown after noticing that

$$\mathbb{P}_p(A)(1-\mathbb{P}_p(A)) \leq \frac{1}{2}\min(\mathbb{P}_p(A), 1-\mathbb{P}_p(A)).$$

Since $\sum_{e\in E} I_A(e) \leq N\max_e I_A(e)$ we have that

$$\max_e I_A(e) \geq \frac{c\mathbb{P}_p(A)(1-\mathbb{P}_p(A))\log\left(\frac{1}{\max_e I_A(e)}\right)}{N} \geq \hat{c}\mathbb{P}_p(A)(1-\mathbb{P}_p(A))\frac{\log(N)}{N}$$

where $0 < \hat{c} \leq \dfrac{\log(1/\max_e I_A(e))}{\log(N)}$.

□

Interestingly, but not useful to us, this discrete theorem has a continuous analogue. The only techniqual detail needed to prove it is the fact that every increasing subset of the cube $[0,1]^N$ is Lebesgue-measurable. The theorem is stated below without proof.

Theorem 2.3.5 *There exists a constant $c \in (0,\infty)$ such that if $|E| = N > 1$ and if $A \subset [0,1]^N$ is an increasing subset with Lebesgue measure in the interval $(0,1)$, then*

$$\sum_{e\in E} I_A(e) \geq c\mathbb{P}_{Leb}(A)(1-\mathbb{P}_{Leb})\log\left(\frac{1}{2\max_e I_A(e)}\right).$$

Furthermore there exists an edge $e \in E$ such that

$$I_A(e) \geq c\mathbb{P}_{Leb}(A)(1-\mathbb{P}_{Leb}(A))\frac{\log(N)}{N}.$$

2.4 Russo's formula

The last section gave us a result that can't be considered as intuitive. Less surprising is the result of this section. It turns out that the change in probability of an increasing event as a function of p is related to the total influence of all the edges on this set.

Theorem 2.4.1 *For any event $A \subset \Omega$*

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(A) = \sum_{e \in E} \mathbb{P}_p(A^e) - \mathbb{P}_p(A_e) = \sum_{e \in E} I_A(e) \tag{2.24}$$

where $\Omega = \{0,1\}^E$ is a finite product space.

Proof. First of all, since Ω is a finite product space

$$\mathbb{P}_p(A) = \sum_{\omega \in \Omega} 1_A(\omega)\mathbb{P}_p(\omega), \qquad A \subset \Omega. \tag{2.25}$$

Furthermore $\mathbb{P}_p$ is a product measure with density p so we have

$$\mathbb{P}_p(\omega) = p^{n(\omega)}(1-p)^{N-n(\omega)}, \tag{2.26}$$

where $n(\omega)$ is the number of open edges in the configuration ω, $n(\omega) = |\{e \in E : \omega(e) = 1\}|$, and N is the number of edges in E, $N = |E|$. In the following calculation we use (2.25) and (2.26) to rewrite the left hand side of (2.24)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(A) &= \frac{\mathrm{d}}{\mathrm{d}p}\left(\sum_{\omega \in \Omega} 1_A(\omega)\mathbb{P}_p(\omega)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}p}\left(\sum_{\omega \in \Omega} 1_A(\omega)p^{n(\omega)}(1-p)^{N-n(\omega)}\right) \\
&= \sum_{\omega \in \Omega} 1_A(\omega)\frac{\mathrm{d}}{\mathrm{d}p}\left(p^{n(\omega)}(1-p)^{N-n(\omega)}\right) \\
&= \sum_{\omega \in \Omega} 1_A(\omega)\left(\frac{n(\omega)}{p}p^{n(\omega)}(1-p)^{N-n(\omega)} - \frac{N-n(\omega)}{p-1}p^{n(\omega)}(1-p)^{N-n(\omega)}\right) \\
&= \sum_{\omega \in \Omega}\left(\frac{n(\omega)}{p} - \frac{N-n(\omega)}{p-1}\right)1_A(\omega)\mathbb{P}_p(\omega). \qquad (2.27)
\end{aligned}$$

What we would like to do is to sum over the edges in our lattice instead of the configurations in our sample space. We need a clever rewriting. To do this we introduce 1_e, the indicator function that e is open. Notice that $\mathbb{P}_p(1_e) = p\ \forall e \in E$ and that $n(\omega) = \sum_{e\in E} 1_e$. In the next step we rewrite (2.27)

$$\begin{aligned}
p(1-p)\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(A) &= \sum_{\omega\in\Omega} (n(\omega)(1-p) - Np + n(\omega)p)\, 1_A(\omega)\mathbb{P}_p(\omega) \\
&= \sum_{\omega\in\Omega} (n(\omega) - Np)\, 1_A(\omega)\mathbb{P}_p(\omega) \\
&= \mathbb{P}_p\left((n(\omega) - Np)1_A(\omega)\right) \\
&= \sum_{e\in E} \mathbb{P}_p(1_e 1_A) - \mathbb{P}_p(1_e)\mathbb{P}_p(1_A). \qquad (2.28)
\end{aligned}$$

Now rewrite the probabilities in (2.28) in terms of A^e and A_e to reach an expression for influence. For clarity, lets examine the probabilities one by one. We have seen earlier in the proof that $\mathbb{P}_p(1_e) = p$. The other two require some calculation

$$\begin{aligned}
\mathbb{P}_p(1_A) &= \mathbb{P}_p\big(\, [\{\omega \in \Omega : \omega(e) = 1\} \cap \{\omega \in \Omega : \omega \in A^e\}] \\
&\quad \cup [\{\omega \in \Omega : \omega(e) = 0\} \cap \{\omega \in \Omega : \omega \in A_e\}]\,\big) \\
&= p\mathbb{P}_p(A^e) + (1-p)\mathbb{P}_p(A_e)
\end{aligned}$$

In the last step we used that all edges are independent.

$$\begin{aligned}
\mathbb{P}_p(1_e 1_A) &= \mathbb{P}_p(\, \{\omega \in \Omega : \omega(e) = 1\} \cap \{\omega \in \Omega : \omega \in A\}\,) \\
&= \mathbb{P}_p(\, \{\omega \in \Omega : \omega(e) = 1\} \cap \{\omega \in \Omega : \omega \in A^e\}\,) \\
&= p\mathbb{P}_p(A^e).
\end{aligned}$$

In the second equality we used that $A = A^e \cup A_e$, but A_e is disjoint from the event that e is open, so only A^e remains. In the last equality we again used that all edges are independent. We plug the expressions for the three probabilities into (2.28)

$$\begin{aligned}
p(1-p)\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(A) &= \sum_{e\in E} p\mathbb{P}_p(A^e) - p(p\mathbb{P}_p(A^e) + (1-p)\mathbb{P}_p(A_e)) \\
&= \sum_{e\in E} p(1-p)\mathbb{P}_p(A^e) - p(1-p)\mathbb{P}_p(A_e).
\end{aligned}$$

The last step is to divide the last equation with $p(1-p)$ on both sides and use the definition of influence. Russo's formula, voila! □

Chapter 3

The main theorem

Before we can prove that the critical probability p_c of $\mathbb{L}^2$ is one half in section 3.3, we need a few more results. In section 3.1 we prove that if there is an infinite open cluster, it is unique. Section 3.1 will be used to give an alternative proof that $p_c \geq 1/2$. It is therefore not necessary for the understanding of the first version of the proof of the main theorem and can be skipped. Section 3.2 though gives us the last results we need to do the first version of the proof of the main theorem.

3.1 Uniqueness of the infinite open cluster

The question about the number of infinite open clusters is not only interesting for us as a tool for doing the alternative proof. If we answer it we will get a deeper understanding of how percolation behaves globally. This is definately not an obvious theorem since there is a configuration $\omega \in \Omega$ for which there are n infinite open clusters, for all $n \in \mathbb{N}$. For example, if we only let the edges on one of the axis be open, we get a configuration with one infinite open cluster. If we open up a parallel line to the open axis, we get a new configuration with two infinite open clusters, and so on.

Theorem 3.1.1 *If $\theta(p) > 0$, then $\mathbb{P}_p(N = 1) = 1$ where $N = N(\omega)$ is the number of infinite open clusters.*

The proof is rather long and an outline is well in place. The proof will be divided into three parts. First some new notation is introduced to ease the calculations. The second step is to rule out all cases where $N = 2, 3, 4, \ldots$. This part is the easy part and follows quickly from the observation that the number of infinite open clusters must be constant. The third step is to rule out the case $N = \infty$. This part has a heavy construction that takes a while to explain, but when it is done the result follows from a contradiction. We will also need a lemma about translation invariant events

Lemma 3.1.2 *If an event is translation invariant, then its probability is either 0 or 1.*

This can be proven with Kolmogorov's 0/1-law. An outline: Let the σ-algebra $\mathcal{F}_n$ be generated by all vertices withing distance n from the origin. A translation invariant event can be appproximated by something in $\mathcal{F}_n$, for a certain n, but also by something in $k + \mathcal{F}_n$. For a large value on k these are independent. Since the event *"there exists N infinite open clusters"* is a translation invariant event, this lemma applies to it.

Proof. Part 1. So first some notation. Given a finite set of vertices B with connecting set of edges edges E_B, let

- $N_B(0)$ be the number of open infinite clusters in the lattice when all edges in E_B are closed.
- $N_B(1)$ be the number of open infinite clusters in the lattice when all edges in E_B are open.
- M_B be the number of open infinite clusters intersecting B.

Two things that will be used later in the proof are important to observe at this stage. If all the edges in B are forced to be open, otherwise disjoint open infinite clusters can be connected. From this the inequality $N_B(0) \geq N_B(1)$ follows. Also, if we enlarge B it will intersect more and more of the infinite open clusters until it intersects all of them, thus by continuity of the probability measure $M_B \to N$ as $B \to \mathbb{Z}^d$.

Part 2. The theorem is trivial for $p = 0$ or $p = 1$, hence let $p \in (0, 1)$. The first thing we shall show is that the number of infinite open clusters is almost surely constant. As earlier, let N be the number of infinite open clusters. Lemma 3.1.2 implies that the event $\{N = k\}, k \in \{0, 1, 2, \ldots, \infty\}$, has probability either 0 or 1 for each k. Since we cant have a different number of infinite open clusters at the same time, the events $\{N = k\}$ are disjoint. Furthermore, in their union we can find all possible configurations of Ω and thus $\bigcup_{k=0}^{\infty}\{N = k\} = \Omega$. Since $\mathbb{P}_p(\Omega) = 1$, we deduce that there is some k for which $\mathbb{P}_p(\{N = k\}) = 1$ and thus the number of infinite open clusters is almost surely constant.

Let $D = D(n)$ be a tilted square with origo as center, $D(n) = \{x \in \mathbb{Z}^d : \delta(0, x) \leq n\}$.

Here δ is the graph theoretic distance explained in chapter 1. Suppose there are k infinite open clusters. The edge set E_D is finite and thus every configuration on E_D has a strictly positive probability. This together with the almost sure constantness of N gives

$$\mathbb{P}_p(N_D(0) = N_D(1) = k) = 1, \tag{3.1}$$

and therefore

$$\mathbb{P}_p(M_D \geq 2) = 0 \ \forall \ D \tag{3.2}$$

If the size of D is increased the number of infinite openn clusters intersecting D will not decrease. Thus M_D is non-decreasing in n and it has a limit in n

$$\lim_{n\to\infty} \mathbb{P}_p(M_{D(n)} \geq 2) \to \mathbb{P}_p(N \geq 2). \tag{3.3}$$

Equation (3.1) and equation (3.3) together implies that Lemma 3.1.2 holds for $k \leq 1$. It remains to rule out the case when $k = \infty$.

Part 3. Assume, towards a contradiction, that Lemma 3.1.2 holds for $k = \infty$. The trick in this part will be to construct a special kind of vertex, called a *trifurcation*. From the assumtion, we can then reach a contradiction about the growth speed of the number of trifurcations. We begin with the definition of a trifurcation.

Definition 3.1.1 *A vertex x is a trifurcation if*

1. *x lies in an infinite open cluster*
2. *x has exactly 3 open edges incident to it*
3. *removing x and its open edges splits the infinite open cluster intro exactly 3 infinite open clusters*

The event that x is a trifurcation is denoted T_x and 1_{T_x} is the indicator function associated with it. By translation-invariance of T_x, $\mathbb{P}_p(T_x)$ is constant for all vertices, and we have

$$\mathbb{P}_p(T_0)|D(n)| = \mathbb{E}_p\left[\sum_{x\in D(n)} 1_{T_x}\right] \tag{3.4}$$

If we can show that $\mathbb{P}_p(T_0) > 0$, (3.4) tells us that the number of trifurcations grow like $|D(n)|$. To do it, we will use our assumption. Let $M_D(0)$ be the number of infinite open clusters that intersect D when all edges in E_D are closed. Note that $M_D(0) \geq M_D$ since $N_D(0) \geq N$. By using the results in the end of Part 2 we can arrive to the limit

$$\mathbb{P}_p(M_{D(n)}(0) \geq 3) \geq \mathbb{P}_p(M_{D(n)} \geq 3) \to \mathbb{P}_p(N \geq 3) = 1, \quad \text{as } n \to \infty \tag{3.5}$$

In the last equality we used the assumption. By monotonicity of $M_D(0)$ we know that there exists an m such that $\mathbb{P}_p(M_{D(n)}(0) \geq 3) \geq 1/2$. Note that $\{M_{D(n)}(0) \geq 3\}$ is independent of the configuration of the edges in $E_{D(n)}$. By the definition of $M_{D(n)}(0)$ it will not change value if anything happens in $E_{D(n)}$. Also, if $\{M_{D(n)}(0) \geq 3\}$ occurs then there exists three vertices $x, y, z \in \partial D(n)$ lying in distinct infinite open clusters of $E \backslash E_{D(n)}$.

Let $\omega \in \Omega$ be a configuration which is also in $\{M_D(0) \geq 3\}$, that is ω gives us three infinite open clusters when all edges in E_D are closed. Pick $x(\omega), y(\omega)$ and $z(\omega)$ according to the previous paragraph.

Lemma 3.1.3 *In $E_{D(m)}$ there exists three paths joining the origin to $x(\omega), y(\omega)$ and $z(\omega)$, and these paths can be chosen so that*

- *the origin is the unique vertex common to any two of the paths*
- *each path touches exactly one vertex in ∂D*

Proof. Let J be the event that all the edges in the paths in Lemma 3.1.3 are open and that all other edges in $E_{D(m)}$ are closed. Since $D(m)$ is finite we can bound the probability that J happens conditioned on $\{M_{D(m)}(0) \geq 3\}$ away from zero.

$$\mathbb{P}_p(J \mid M_{D(m)}(0) \geq 3) \geq (\min(p, 1-p))^{|E_{D(m)}|} > 0.$$

If we multiply the left hand side with $\mathbb{P}_p(M_{D(m)}(0) \geq 3)$, by the definition of conditional probability we get that

$$\begin{aligned}\mathbb{P}_p(J \cap \{M_{D(m)}(0) \geq 3\}) &= \mathbb{P}_p(J \mid M_{D(m)}(0) \geq 3)\mathbb{P}_p(M_{D(m)}(0) \geq 3) \\ &\geq \frac{1}{2}(\min(p, 1-p))^{|E_{D(m)}|} > 0.\end{aligned}$$

But the event $J \cap \{M_{D(m)}(0) \geq 3\}$ is a trifurcation, since we can choose the paths in Lemma 3.1.3 as we like. There are more than one choice for a general m and therefore the probability that the origin is a trifurcation has positive probability

$$\mathbb{P}_p(T_0) \geq \mathbb{P}_p(J \cap \{M_{D(m)}(0) \geq 3\}) > 0.$$

□

We conclude from (3.4) and Lemma 3.1.3 that the number of trifurcations must grow like $|D(n)|$ as $n \to \infty$. But from the definition of a trifurcation we can quickly arrive to the conclusion that the number of trifurcations must be bounded by the number of elements in the boundary of $D(n)$. Select a trifurcation in $D(n)$, say t_1, and choose a vertex $v_1 \in \partial D(n)$ such that $t_1 \leftrightarrow v_1 \in D(n)$. Select a second trifurcation $t_2 \in D(n)$. The definition of a trifurcation says that if we remove the trifurcation and its open paths, then the infinite open cluster it lies in must split into three infinite open clusters. Therefore, there must exists a vertex $v_2 \in \partial D(n)$ such that $t_2 \leftrightarrow v_2 \in D(n)$ and $v_1 \neq v_2$. This process can be continued for all trifurcations in $D(n)$, but we see that since they all need to be connected to a vertex in $\partial D(n)$ that no other trifurcation is connected to, the number of trifurcations is bounded by $|\partial D(n)|$.

Now $|\partial D(n)|$ grows like n^{d-1} as n increases, and the number of trifurcations is bounded by this growth. But at the same time we showed that since $\mathbb{P}_p(T_0) > 0$ the number of trifurcations also grows like $|D(n)|$, which is of order n^d. This contradicts that $k = \infty$. We are left with the options $k = 0$ or $k = 1$ and hence the infinite open cluster is unique.

□

3.2 Crossing a rectangle

The crucial tool in the proof of the main theorem will be probabilities that rectangles have open horizontal or vertical paths that cross them. We will denote an $m \times n$ rectangle by $R_{m,n}$. Every rectangle $R_{m,n}$ has a horizontal dual $R^h_{m,n}$ and a vertical dual $R^v_{m,n}$. For $m, n \geq 2$ the horizontal dual $R^h_{m,n}$ is a $m-1 \times l+1$ rectangle, and the vertical dual $R^v_{m,n}$ is a $m+1 \times l-1$ rectangle.

Two important events in the proof of the main theorem concerns open paths that cross $R_{m,n}$ horizontally or vertically. For an $\omega \in \Omega$, an open horizontal crossing of $R_{m,n}$ is an open path joining one vertex on the left boundary of $R_{m,n}$ to one vertex on the right boundary. The standard notation for the event that $R_{m,n}$ has such a crossing is $H(R_{m,n})$. The event that $R_{m,n}$ has a vertical crossing is defined analogously and denoted $V(R_{m,n})$.

The following lemma is in some sense obvious. The proof is omitted, but can be found in [6].

Lemma 3.2.1 *Let $R_{m,n}$ be a rectangle in the square lattice or its dual. For all configurations of the bonds in $R_{m,n}$, exactly one of the events $H(R_{m,n})$ and $V(R^h_{m,n})$ happens.*

So the lemma states that either we have an open horizontal crossing of the rectangle, or we have an open vertical crossing of its dual. This makes much sense, since if there is no open horizontal crossing, then there must be a closed vertical crossing cutting off any open horizontal paths that "tries" to reach the other side of the rectangle. But this corresponds to an open vertical crossing of the dual.

From Lemma 3.2.1 we can deduce a couple of very useful relations about crossing probabilities. They are useful because they give us a quantative result. First of, we have

$$\mathbb{P}_p(H(R_{k,l-1})) + \mathbb{P}_{1-p}(V(R_{k-1,l})) = 1. \tag{3.6}$$

Lemma 3.2.1 tells us that every configuration is in exactly one of $\mathbb{P}_p(H(R_{k,l-1}))$ and $\mathbb{P}_p(H(R^h_{k,l-1}))$. But $R^h_{k,l-1}$ is a $k-1 \times l$ rectangle in the dual of $\mathbb{L}^2$. Here edges are open with probability $1-p$. Therefore $\mathbb{P}_p(H(R^h_{k,l-1})) = \mathbb{P}_{1-p}(V(R_{k-1,l}))$ and (3.6) follows. Secondly, we have

$$\mathbb{P}_{\frac{1}{2}}(H(R_{n+1,n})) = 1/2 \tag{3.7}$$

because of the symmetry $\mathbb{P}_{\frac{1}{2}}(H(R_{n+1,n})) = \mathbb{P}_{\frac{1}{2}}(V(R_{n,n+1}))$. Finally,

$$\mathbb{P}_{\frac{1}{2}}(H(R_{n,n})) = \mathbb{P}_{\frac{1}{2}}(V(R_{n,n})) \geq 1/2 \tag{3.8}$$

since $H(R_{n,n-1}) \subset H(R_{n,n})$ and the probability measure is monotonic.

What we would like to do next is to connect rectangles, and find out what the probability is that the (not disjoint) union of two rectangles has an open crossing. The event $X(R)$ will help us do this.

Definition 3.2.1 *Let $R_{m,2n}$, $m \geq n$ and $R_{n,n}$ have their lower left corner in origo. $X(R_{m,2n})$ is the event that there exists two open paths $P_1 \subset R_{n,n}$ and $P_2 \subset R_{m,2n}$ such that P_1 crosses $R_{n,n}$ from top to bottom and P_2 connects P_1 with the right boundary of $R_{m,2n}$.*

The next proposition tells us that we can bound the probability of $X(R)$ from below with probabilities we already know. The proposition is crucial for connecting rectangles.

Proposition 3.2.2

$$\mathbb{P}_p(X(R_{m,2n})) \geq \mathbb{P}_p(H(R_{m,2n}))\mathbb{P}_p(V(R_{n,n}))/2 \tag{3.9}$$

Proof. First we do a short construction, but most of the proof is calculation.

Suppose that there is an open path P from top to bottom of $R_{n,n}$, i.e. that $V(R_{n,n})$ happens. There may be multiple such P, and we would like to choose one of them. Let $LV(R_{n,n})$ be the left most such P, say P_1. Note that, since all edges are independent, the event $\{LV(R_{n,n}) = P_1\}$ is independent of all edges to the right of P_1. The construction is done, and now follows some calculations that lead us to the result.

Lemma 3.2.3 *For all possible values P_1 of $LV(R_{n,n})$*

$$\mathbb{P}_p\left(X(R_{m,2n}) \mid LV(R_{n,n}) = P_1\right) \geq \mathbb{P}_p\left(H(R_{m,2n})\right)/2. \tag{3.10}$$

Proof. Let P_1 be reflected in the horizontal axis of symmetry in $R_{m,2n}$. We will then get a second path P_1'. Add one edge e if needed to connnect P_1 with P_1'. Now $P' = P_1 \cup \{e\} \cup P_1'$ is a vertical crossing of $R_{m,2n}$. If we have an open horizontal crossing of $R_{m,2n}$, say P_3, this path must cross P' at some vertex. The probability that P_3 exists is $\mathbb{P}_p(H(R_{m,2n}))$. By the symmetry of P, the probability that some path like P_3 meets P at a vertex in P_1 is $\mathbb{P}_p(H(R_{m,2n}))/2$.

Now define $Y(P_1)$ to be the event that there exists an open path $P_2 \subset R_{m,2n}$ to the right of P joining P_1 with the right boundary of $R_{m,2n}$. Note that the event that P_3 meets P at a vertex in P_1 is a subset of $Y(P_1)$, and hence

$$\mathbb{P}_p(Y(P_1)) \geq \mathbb{P}_p\left(H(R_{m,2n})\right)/2.$$

Also note that $Y(P_1)$ only depends on bonds to the right of P_1, hence $Y(P_1)$ and $LV(R_{n,n}) = P_1$ are independent. Thus

$$\mathbb{P}_p\left(Y(P_1) \mid LV(R_{n,n})\right) = \mathbb{P}_p(Y(P_1)) \geq \mathbb{P}_p(H(R_{m,2n}))/2.$$

But if $Y(P_1)$ holds and the leftmost open vertical crossing of $R_{n,n}$ is P_1, then by definition $X(R_{m,2n})$ holds,

$$\mathbb{P}_p\left(X(R_{m,2n}) \mid LV(R_{n,n}) = P_1\right) \geq \mathbb{P}_p(H(R_{m,2n}))/2.$$

□

But as $V(R_{n,n})$ is the disjoin union of all $LV(R_{n,n}) = P_i$ we get

$$\begin{aligned} \mathbb{P}_p\left(X(R_{m,2n}) \mid V(R_{n,n})\right) &= \frac{\mathbb{P}_p\left(X(R_{m,2n} \cap V(R_{n,n})\right)}{\mathbb{P}_p(V(R_{n,n}))} \geq \mathbb{P}_p(H(R_{m,2n}))/2 \\ \Rightarrow \mathbb{P}_p(X(R_{m,2n})) &\geq \mathbb{P}_p(H(R_{m,2n}))\mathbb{P}_p(V(R_{n,n}))/2. \end{aligned}$$

□

The strength of this proposition is, as stated earlier, that it helps us find lower bounds for crossing probabilities. We will conclude this section with finding two such bounds. It will not only be done to demonstrate Proposition 3.2.2, but they will also be used in the proof of the main theorem. First we need an intermediate result

Corollary *For all $n \geq 1, \mathbb{P}_{\frac{1}{2}}\left(H(R_{3n,2n})\right) \geq 2^{-7}$.*

Proof. Let $R_{2n,2n}$ and $R'_{2n,2n}$ overlap halfway, so that $R_{2n,2n} \cap R'_{2n,2n}$ is a $n \times 2n$ rectangle. Note that $R_{2n,2n} \cup R'_{2n,2n}$ now is a $3n \times 2n$ rectangle, exacly what we want to examine. Call the lower $n \times n$ part of the intersection S. This square is where we have a vertical crossing if $X(R_{2n,2n})$ happens. Let $X'(R'_{2n,2n})$ be defined in the same way as $X(R'_{2n,2n})$ but reflected horizontally, so that if $X'(R'_{2n,2n})$ happens the vertical crossing also is in S. Proposition 3.2.2 now tells us that,

$$\mathbb{P}_p\left(X'(R'_{2n,2n})\right) = \mathbb{P}_p\left(X(R'_{2n,2n})\right) \geq \mathbb{P}_p\left(H(R_{2n,2n})\right)\mathbb{P}_p\left(V(S)\right)/2.$$

From Chapter 2 we know that $X(R_{2n,2n})$, $H(R_{2n,2n})$ and $H(S)$ are increasing events. In the following calculation, we use this fact when we apply the FKG inequality.

$$\begin{aligned}
\mathbb{P}_{\frac{1}{2}}\left(H(R_{3n,2n})\right) &= \mathbb{P}_{\frac{1}{2}}\left(H(R_{2n,2n} \cup R'_{2n,2n})\right) \\
&\geq \mathbb{P}_{\frac{1}{2}}\left(X'(R'_{2n,2n}) \cap X(R_{2n,2n}) \cap H(S)\right) \\
&\geq \mathbb{P}_{\frac{1}{2}}\left(X'(R'_{2n,2n}\right)\mathbb{P}_{\frac{1}{2}}\left(X(R_{2n,2n})\right)\mathbb{P}_{\frac{1}{2}}\left(H(S)\right) \\
&\geq \mathbb{P}_{\frac{1}{2}}\left(H(R_{2n,2n})\right)^2\mathbb{P}_{\frac{1}{2}}\left(V(S)\right)^2\mathbb{P}_{\frac{1}{2}}\left(H(S)\right)/4.
\end{aligned}$$

From (3.8) we know that the crossing probability of a square is at least $1/2$, and this gives us the result.

$$\mathbb{P}_{\frac{1}{2}}\left(H(R_{3n,2n3})\right) \geq \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = 2^{-7}.$$

□

We move on to the promised lower bounds. The first one shows that we can we can connect any number of similar rectangles such that their intersection is a square. Consider $R_{m_1,2n}$ and $R_{m_2,2n}$ with $m_1, m_2 \geq 2n$ such that $R_{m_1,2n} \cap R_{m_2,2n}$ is a $2n \times 2n$ squre. Call this square S. Using the FKG inequality and Proposition 3.2.2 we get

$$\begin{aligned}
\mathbb{P}_{\frac{1}{2}}\left(H(R_{m_1+m_2-2n,2n})\right) &= \mathbb{P}_{\frac{1}{2}}\left(H(R_{m_1,2n} \cup H(R_{m_2,2n})\right) \\
&\geq \mathbb{P}_{\frac{1}{2}}\left(H(R_{m_1,2n})\right)\mathbb{P}_{\frac{1}{2}}\left(H(R_{m_2,2n})\right)\mathbb{P}_{\frac{1}{2}}\left(V(S)\right) \\
&\geq \mathbb{P}_{\frac{1}{2}}\left(H(R_{m_1,2n})\right)\mathbb{P}_{\frac{1}{2}}\left(H(R_{m_2,2n})\right)/2.
\end{aligned}$$

This calculation leads us to the next crossing probability,

$$\begin{aligned}
\mathbb{P}_{\frac{1}{2}}\left(H(R_{m+n,2n})\right) &= \mathbb{P}_{\frac{1}{2}}\left(H(R_{m+3n-2n,2n})\right) \\
&\geq \mathbb{P}_{\frac{1}{2}}\left(H(R_{m,2n})\right)\mathbb{P}_{\frac{1}{2}}\left(H(R_{3n,2n})\right)/2 \\
&\geq \mathbb{P}_{\frac{1}{2}}\left(H(R_{m,2n})\right)2^{-8},
\end{aligned}$$

which leads us further on. For $k \geq 3$ and $n \geq 1$,

$$\begin{aligned}\mathbb{P}_{\frac{1}{2}}\left(H(R_{kn,2n})\right) &\geq \mathbb{P}_{\frac{1}{2}}\left(H(R_{(k-1)n,2n})\right) \\ \dots &\geq \mathbb{P}_{\frac{1}{2}}\left(H(R_{3n,2n})\right)2^{-8(k-3)} \\ &\geq 2^{-7}2^{-8(k-3)} = 2^{17-8k}.\end{aligned}$$

Finally, since $H(R_{m,2n}) \subset H(R_{m,2n+1})$, we get the nice result,

$$\text{For each } k \geq 2\ \exists \text{ constant } k_H > 0 \text{ such that } \mathbb{P}_{\frac{1}{2}}\left(H(R_{kn,n})\right) \geq k_H \forall n \geq 1. \tag{3.11}$$

The second bound will be for $6n \times 2n$ rectangles. Consider the same situation as in the previous bound, then

$$\begin{aligned}\mathbb{P}_{\frac{1}{2}}\left(H(R_{m_1+m_2-n,2n})\right) &= \mathbb{P}_{\frac{1}{2}}\left(H(R_{m_1+m_2+n-2n,2n})\right) \\ &\geq \mathbb{P}_{\frac{1}{2}}\left(H(R_{m_1,2n})\right)\mathbb{P}_{\frac{1}{2}}\left(H(R_{m_2+n,2n})\right)/2 \\ &\geq \mathbb{P}_{\frac{1}{2}}\left(H(R_{m_1,2n})\right)\mathbb{P}_{\frac{1}{2}}\left(H(R_{m_2,2n})\right)2^{-9},\end{aligned}$$

$$\begin{aligned}\Rightarrow \mathbb{P}_{\frac{1}{2}}\left(H(R_{5n,2n})\right) &\geq \mathbb{P}_{\frac{1}{2}}\left(H(R_{3n,2n})\right)^2 2^{-9} \\ &\geq 2^{-14-9} = 2^{-23},\end{aligned}$$

$$\begin{aligned}\Rightarrow \mathbb{P}_{\frac{1}{2}}\left(H(R_{6n,2n})\right) &\geq \mathbb{P}_{\frac{1}{2}}\left(H(R_{5n,2n})\right)\mathbb{P}_{\frac{1}{2}}\left(H(R_{2n,2n})\right)2^{-9} \\ &\geq 2^{-23-1-9} = 2^{-33}.\end{aligned} \tag{3.12}$$

3.3 The main theorem

$\mathbb{L}^2$ is one of the lattices for which p_c has been calculated. The fact that $p_c = 1/2$ was an open question for more than 20 years. Harris [3] proved in 1960 that $\theta(1/2) = 0$, and therefore $p_c \geq 1/2$. In 1980 Kesten [7] proved the complementary inequality $p_c \leq 1/2$ and the problem was solved. This classical proof will be presented, but also an alternative way to prove that $p_c \geq 1/2$ will be presented.

Theorem 3.3.1 *On $\mathbb{L}^2$, $p_c = 1/2$ and $\theta(1/2) = 0$.*

Proof. All our previous work (FKG, influence, Russo's formula) will be used in the proof, and it will be split in two parts. In the first part we show that the probability that there is a large open cluster around the origin decays fast when the size of the cluster is increased. This will be done by constructing closed cycles around the origin. It is then easy to deduce that $\theta(1/2) = 0$ from this. In the second part, influence and Russo's formula is used to show that the probability that there is an infinite open cluster when $p > 1/2$ is greater than zero.

Part 1. As a notion of the size of an open cluster, we use the graph theoretic radius

$$r(C) = \sup\{\delta(0, x) : x \in C\}.$$

As promised above, we shall show is that for some positive constant c such that

$$\mathbb{P}_{\frac{1}{2}}(r(C) \geq n) \leq n^{-c}. \tag{3.13}$$

This will come easily from our previous hard work on crossings of rectangles. At $p = 1/2$, it follows from the self duality of $\mathbb{L}^2$ that the edges in $\mathbb{L}^2_d$ are open with probability $1/2$ and independent of each other. Hence (3.12) applies to $6n \times 2n$ -rectangles in $\mathbb{L}^2_d$. Consider the situation where we place two $6n \times 2n$ rectangles, R and R', and two $2n \times 6n$ rectangles, R'' and R''' so that they form a $6n \times 6n$ square with a $2n \times 2n$ hole in the middle. Using the FKG inequality and (3.12), the probability that there exists a dual cycle in the 4 rectangles is

$$\begin{aligned}
\mathbb{P}_{\frac{1}{2}}\left(H(R_{6n,2n}) \cap H(R'_{6n,2n}) \cap V(R''_{2n,6n}) \cap V(R'''_{2n,6n})\right) &\geq \mathbb{P}_{\frac{1}{2}}(H(R_{6n,2n}))^4 \\
&\geq \left(2^{-33}\right)^4 \\
&= 2^{-132} > 0.
\end{aligned}$$

What we do next is construct events that these open dual cycles exist around the origin, and use them to deduce (3.13). Let $A_k, k \geq 1$, be the square made up by four rectangles as above, centered on $(1/2, 1/2)$ with inner radius 4^k and outer radius $3 \cdot 4^k$. Note that all A_k are disjoint sets. Define B_k to be the event that there exists an open dual cycle inside A_k. By the calculation above

$$\mathbb{P}_{\frac{1}{2}}(B_k) \geq 2^{-132}, \quad \forall\, k \geq 1$$

Now note two things about B_k:

- Since all A_k are disjoint and all edges in $\mathbb{L}^2_d$ are independent, all E_k are independent.

- If B_k holds, then there are no open paths through connecting points in $R_{3 \cdot 4^k, 3 \cdot 4^k} \backslash A_k$, that is in the $4^k \times 4^k$ hole inside A_k, with points in $\Omega \backslash R_{3 \cdot 4^k, 3 \cdot 4^k}$, that is outside the outer radius of A_k. Hence we have

$$r(C) \leq 3 \cdot 4^k < 4^{k+1}.$$

With these facts about B_k, we get

$$\begin{aligned}\mathbb{P}_{\frac{1}{2}}\left(r(C) \leq 4^{l+1}\right) &\leq \mathbb{P}_{\frac{1}{2}}\left(\bigcap_{k=1}^{l} E_k^C\right) \\ &= \prod_{k=1}^{l} \mathbb{P}_{\frac{1}{2}}\left(E_k^C\right) \leq \left(1-2^{-132}\right)^l.\end{aligned}$$

The result follows from substituting $n = 4^{l+1}$. From this we can also conclude that $\theta(1/2) = 0$:

$$\theta(1/2) = \mathbb{P}_{\frac{1}{2}}\left(r(C) = \infty\right) \leq \mathbb{P}_{\frac{1}{2}}\left(r(C) \geq n\right) \leq n^{-c} \to 0 \text{ as } n \to \infty. \tag{3.14}$$

Part 2. The result from part one implies that $p_c \geq 1/2$. In this part we build further on the rectangle arguments to finally conclude that $p_c \leq 1/2$. To do this, we need to add two lemmas to the results of Part 1. The first one gives us a lower bound for the influence that one bond in a rectangle can have. The second one gives us a lower bound for $\mathbb{P}_p\left(H(R_{m,n})\right)$.

Let $I_A^p(e)$ be the influence of e on A under $\mathbb{P}_p$.

Lemma 3.3.2 *For $0 < p < 1$ and $e \in R_{m,n}$,*

$$I^p_{H(R_{m,n})}(e) \leq 2\mathbb{P}_{\frac{1}{2}}\left(r(C) \geq \min\left(\frac{m}{2} - 1, \frac{n-1}{2}\right)\right). \tag{3.15}$$

Proof. The proof is really straight forward. Recall that for an increasing set A, the influence of one edge is

$$I_A^p(e) = \mathbb{P}_p\left(A^e\right) - \mathbb{P}_p\left(A_e\right).$$

Assume that $I^p_{H(R_{m,n})}(e) > 0$, the terminology in the litterature for this is that e is pivotial for $H(R_{m,n})$. As $H(R_{m,n})$ is increasing and e is pivotial, the configuration ω^e is in $H(R_{m,n})$ and there will be an open horizontal crossing of $R_{m,n}$. Also ω_e is not in $H(R_{m,n})$ so all open horizontal crossings of $R_{m,n}$ must use e. This tells us that in ω, one endpoint of e is connected with an open path inside $R_{m,n}$ to the left boundary of $R_{m,n}$, and the other endpoint to the left boundary. One of these open paths must have length at least $m/2 - 1$ and thus

$$I^p_{H(R_{m,n})}(e) \leq 2\mathbb{P}_p\left(r(C) \geq \frac{m}{2} - 1\right). \tag{3.16}$$

From Lemma 3.2.1, we know that ω_e is in $V(R^h_{m,n})$. Recall how the horizontal dual of a rectangle was defined, and then it should be obvious by the same argument as above that

$$I^p_{H(R_{m,n})}(e) \leq 2\mathbb{P}_{1-p}\left(r(C) \geq \frac{n-1}{2}\right). \tag{3.17}$$

□

Lemma 3.3.3 *Fix $p > 1/2$ and an integer $\rho > 1$. Then there exists constants $\gamma = \gamma(p) > 0$ and $n_0 = n_0(p, \rho)$ such that*

$$\mathbb{P}_p\left(H(R_{\rho n,n})\right) \geq 1 - n^{\gamma} \tag{3.18}$$

for all $n \geq n_0$.

Proof. We know from Proposition 3.14 and Lemma 3.3.2 that we can bound the influence of $e \in E$ on $H(R_{\rho n,n})$ for $n \geq 2$ and $\hat{p} \in [1/2, p]$

$$I^{\hat{p}}_{H(R_{\rho n,n})}(e) \leq n^{-a}.$$

Now our main result about influence, Theorem 2.3.2, yeilds

$$\sum_{e \in H(R_{\rho n,n})} I^{\hat{p}}_{H(R_{\rho n,n})}(e) \geq c\mathbb{P}_{\hat{p}}(1 - \mathbb{P}_{\hat{p}}) \log(n^a). \tag{3.19}$$

The next step is a trick. Let $g(p) = \log\left(\mathbb{P}_{\hat{p}}/(1 - \mathbb{P}_{\hat{p}})\right)$. We know from Russo's Formula that the sum in (3.19) is equal to the derivative of the probability measure w.r.t. $\hat{p}$. We get

$$\frac{\mathrm{d}}{\mathrm{d}\hat{p}} g(\hat{p}) = \frac{1}{\mathbb{P}_{\hat{p}}(1 - \mathbb{P}_{\hat{p}})} \frac{\mathrm{d}}{\mathrm{d}\hat{p}} \mathbb{P}_{\hat{p}} \geq ac\log(n)$$

The lower bound in (3.11) allows us to integrate this to

$$g(p) \geq ac(p - 1/2)\log(n), \quad \forall\, n \geq n_0(p, \rho),$$

and substituting back for $g(p)$ we get the desired result.

□

Fix $p > 1/2$. Let

$$R_k = \begin{cases} R_{2^k n, 2^{k+1} n}, & k \text{ is even,} \\ R_{2^{k+1} n, 2^k n}, & k \text{ is odd,} \end{cases}$$

and place them so that all R_k have their lower left corner in origo. Let E_k be the event that R_k has an open crossing in the long direction. In this construction, an open crossing of R_k will intersect an open crossing of R_{k+1} if they both happen. This implies that if all E_k happens, then E_∞ happens. By choosing a large enough n, we can apply Lemma 3.3.3

$$\begin{aligned}
\mathbb{P}_p(E_\infty) &\geq \mathbb{P}_p\left(\bigcap_{k=0}^{\infty} E_k\right) \\
&= 1 - \mathbb{P}_p\left(\bigcup_{k=0}^{\infty} E_k^C\right) \\
&= 1 - \sum_{k=0}^{infty} E_k^C \\
&\geq 1 - \sum_{k=0}^{\infty}(2^k n)^{-\gamma} \\
&= 1 - \frac{n^\gamma}{1 - 2^{-\gamma}} = 1 - \frac{1}{n^\gamma - (n/2)^\gamma} > 0.
\end{aligned}$$

Since E_∞ has positive probability of happening if $p > 1/2$, by definition

$$p_c \leq 1/2. \tag{3.20}$$

From (3.14) and (3.20), we conclude that the value for p_c must be $1/2$ for bond percolation on the square lattice.

□

Alternative proof of Part 1. Let $p = 1/2$ and assume towards a contradiction that $\theta(1/2) > 0$. Let $T(n) = [0, n]^2$ be the square with origo in its lower left corner and side length n. Suppose that there is a vertex in $T(n)$ that is part of an infinite open cluster. If this is the case, then there must be an open path from at least one vertex on the boundary $\partial T(n)$ to infinity. We conclude that if a vertex in $T(n)$ is a part on an infinite open cluster, then there is a vertex in $\partial T(n)$ from which there is an open path to infinity that doesn't include another vertex from $T(n)$.

With this as motivation, we define A^l to be event that there is an open path of infinite length starting from the left boundary of $T(n)$. For the right, top and bottom boundary we define A^r, A^t and A^b in the same way. Saying that $T(n)$ is intersecting an infinite open cluster is now equivalent to saying that the event $A^l \cup A^r \cup A^t \cup A^b$ happens. By translation invariance

$$\mathbb{P}_{\frac{1}{2}}(A^l) = \mathbb{P}_{\frac{1}{2}}(A^r) = \mathbb{P}_{\frac{1}{2}}(A^t) = \mathbb{P}_{\frac{1}{2}}(A^b).$$

Observe that our assumption $\theta(1/2) > 0$ gives every vertex in $T(n)$ a positive probability of being a part in an infinite open cluster. For a general n, the FKG inequality gives us

$$\begin{aligned}\mathbb{P}_{\frac{1}{2}}(T(n) \nleftrightarrow \infty) &= \mathbb{P}_{\frac{1}{2}}\left((A^l)^C \cap (A^r)^C \cap (A^t)^C \cap (A^b)^C\right) \\ &\geq \mathbb{P}_{\frac{1}{2}}\left((A^l)^C\right)\mathbb{P}_{\frac{1}{2}}\left((A^r)^C\right)\mathbb{P}_{\frac{1}{2}}\left((A^t)^C\right)\mathbb{P}_{\frac{1}{2}}\left((A^b)^C\right) \\ &= \mathbb{P}_{\frac{1}{2}}\left((A^l)^C\right)^4. \end{aligned} \tag{3.21}$$

Pick a large N such that

$$\mathbb{P}_{\frac{1}{2}}\left(\partial T(N) \leftrightarrow \infty\right) > 1 - \epsilon, \quad \epsilon > 0. \tag{3.22}$$

Inserting (3.22) into (3.21), and doing some rewriting, yields

$$\mathbb{P}_{\frac{1}{2}}\left(A^l\right) \geq 1 - \mathbb{P}_{\frac{1}{2}}\left(T(n) \nleftrightarrow \infty\right)^{1/4} > 1 - \epsilon^{1/4}. \tag{3.23}$$

We move on to study the same situation for the dual graph, $T^*(n) = [0, n]^2 + \left(\frac{1}{2}, \frac{1}{2}\right)$. For the dual graph, let the event A_d^l be that there exists an infinite closed path from a vertex on the left boundary of $T^*(n)$ which doesnt include another vertex from $T^*(n)$. The events A_d^r, A_d^t and A_d^b are defined in the same way. Since the probability that an edge is closed is $1/2$, we can from the same arguments as for the original graph deduce that

$$\mathbb{P}_{\frac{1}{2}}(A_d^l) > 1 - \epsilon^{1/4}, \tag{3.24}$$

but we may have to choose a bigger n to work with, lets say N_d.

So far we have only made a construction and explored the consequences of it. Now comes the trick in this part of the proof. For $n = \max(N, N_d)$, consider the event $A = A^l \cap A^r \cap A_b^t \cap A_d^b$. It is time to choose a value for ϵ. From (3.22) and (3.24) we get

$$\begin{aligned}\mathbb{P}_{\frac{1}{2}}\left(A^C\right) &= \mathbb{P}_{\frac{1}{2}}\left((A^l)^C \cup (A^r)^C \cup (A_d^t)^C \cup (A_d^b)^C\right) \\ &\leq \mathbb{P}_{\frac{1}{2}}\left((A^l)^C\right) + \mathbb{P}_{\frac{1}{2}}\left((A^r)^C\right) + \mathbb{P}_{\frac{1}{2}}\left((A_d^t)^C\right) + \mathbb{P}_{\frac{1}{2}}\left((A_d^b)^C\right) \\ &\leq 4\epsilon^{1/4}.\end{aligned}$$

By choosing $\epsilon = (1/8)^4$, we get $\mathbb{P}_{\frac{1}{2}}(A^C) \leq 1/2$ and $\mathbb{P}_{\frac{1}{2}}(A) \geq 1/2$.

If A occurs, we have open paths of infinite length from the left and right boundary of $T(N)$ and we have closed paths from the top and bottom boundary of $T^*(N)$. But by theorem 3.1.1, $\mathbb{L}^2$ can have at most one infinite open cluster. This means that the open paths of infinite length must be connected inside $T(N)$. But this makes it impossible to connect the closed paths of infinite length inside $T^*(N)$. Since the $\mathbb{L}^2$ is self-dual, this contradicts the uniqueness of the infinite open cluster. We conclude that the event A must have probability 0. This contradicts our result that $\mathbb{P}_{\frac{1}{2}}(A) \geq 1/2$ and the assumption $\theta(1/2) > 0$ is false. Therefore $\theta(1/2) = 0$ and by definition

$$p_c \geq \frac{1}{2}. \tag{3.25}$$

□

Chapter 4

Appendix

4.1 Cylinder sets

Our usual setting to work with in this report is to have a countably infinite sample space $\Omega = \{0, 1\}^{\mathbb{N}}$. Let, by analogy, the state of each edge be represented by a coin toss. This is a very fair analogy if we let all edges be open with probability $p = 1/2$, or closed otherwise. Since Ω is countably finite, it makes sense to order the edges (coin tosses). Now, for whatever event we are interested in, we can instead ask ourselves *"what is the chance that the sixth, ninth, eleventh, ect... coin tosses are heads?"* This event is by our definition a cylinder set, and something we would like to measure.

4.2 Proof of the hypercontractivity theorem

We will here prove a slightly more general version than the one stated in Chapter 2, but by choosing $q = 2$ we get the same result.

Theorem 4.2.1 *Let $1 \leq p \leq q \leq \infty$, and let $\rho \leq \sqrt{\dfrac{p-1}{q-1}}$.*
Then for all $f : -1, 1^n \to \mathbb{R}$ we have

$$||T_\rho f||_q \leq ||f||_p.$$

Proof. The proof will be done in two steps. In the first step we consider the case when $n = 1$ and construct a "two point inequality", since the inequality only can depend on the two values $f(1)$ and $f(-1)$. The second step is to extend the result for arbitrary n by induction on n.

Step 1. As stated above, if $n = 1$ then $f : \{-1, 1\} \to \mathbb{R}$ can be represented by the two values $a = f(1)$ and $b = f(-1)$. If we apply T_ρ, defined by (2.18), we get

$$T_\rho f(1) = \left(\frac{1+\rho}{2}\right) a + \left(\frac{1-\rho}{2}\right) b, \quad T_\rho f(-1) = \left(\frac{1-\rho}{2}\right) a + \left(\frac{1+\rho}{2}\right) b.$$

Think of all functions $f : \{-1, 1\} \to \mathbb{R}$ as points $(a, b) \in \mathbb{R}^2$, in which case $T_\rho f$ are line segments that joins (a, b) to (b, a). Note that when $\rho = 1$, $T_\rho f = f$. Also note that when $\rho \to 0$, $T_\rho f$ moves towards the midpoint of the line joining (a, b) with (b, a).

So, by identifying Boolean functions with points in $\mathbb{R}^2$, we can reformulate the problem to the following. Given a and b, what is the largest $\rho = \rho(p, q)$ such that

$$\|T_\rho(a, b)\|_q \leq \|(a, b)\|_p.$$

Many points (functions) have the same norm $\|(a, b)\|_p$. This set of points is can be though of as level set in l_p, i.e. an l_p-sphere

$$\left\{(a, b) : \left(\frac{|a|^p + |b|^p}{2}\right)^{1/p} = \text{constant}\right\}.$$

For $p = 2$, we recognize these level sets as circles. For $p > 2$ they look more and more like squares. For $p < 2$ they look more and more like diamonds. So restating the problem again, we ask "How far towards the middle of $T_\rho(a, b)$ do we have to go to get inside the l_q curve?". The worst case, i.e. the one that is most constraining for ρ, is when a and b are close together. Without loss of generality, let $a = 1 + \epsilon$ and $b = 1 - \epsilon$ for some small $\epsilon > 0$. Now we find the smallest possible ρ so that

$$\begin{aligned} & \|T_\rho(1+\epsilon, 1-\epsilon)\|_q & \leq & \; \|(1+\epsilon, 1-\epsilon)\|_p \\ \iff & \|(1+\rho\epsilon, 1-\rho\epsilon)\|_q & \leq & \; \|(1+\epsilon, 1-\epsilon)\|_p \\ \iff & \left(\frac{(1+\rho\epsilon)^q + (1-\rho\epsilon)^q}{2}\right)^{1/q} & \leq & \; \left(\frac{(1+\epsilon)^p + (1-\epsilon)^p}{2}\right)^{1/p}. \end{aligned} \tag{4.1}$$

From the binomail theorem we know that

$$(1+\rho\epsilon)^q = 1 + q\rho\epsilon + \frac{q(q-1)}{2}\rho^2\epsilon^2 + \dots.$$

Applying this on the fraction in the right hand side of (4.1) makes all odd terms dissapear

$$\frac{(1+\rho\epsilon)^q + (1-\rho\epsilon)^q}{2} = 1 + \frac{q(q-1)}{2}\rho^2\epsilon^2 + O(\epsilon^4). \tag{4.2}$$

Furthermore,

$$(1+\delta)^q = 1 + q\delta + O(\delta^2). \tag{4.3}$$

Hence by applying (4.2) and (4.3) to (4.1) we get

$$\text{LHS} = 1 + \frac{q-1}{2}\rho^2\epsilon^2 + O(\epsilon^4), \quad \text{RHS} = 1 + \frac{p-1}{2}\epsilon^2 + O(\epsilon^4). \tag{4.4}$$

Now if LHS ≤ RHS as $\epsilon \to 0$ we need to have

$$\frac{q-1}{2}\rho^2 \leq \frac{p-1}{2} \Rightarrow \rho \leq \sqrt{\frac{p-1}{q-1}},$$

and the theorem is proven for the case $n = 1$.

Step 2. The induction will be done slightly differently than how it is usually done. Instead of assuming that the theorem holds for $n = m$, we will make a partition of n. Let the coordinates $[n]$ be partitioned into I and J, and we will write $\{-1, 1\}$ as (x, y) where $x \in \{-1, 1\}^I$ and $y \in \{-1, 1\}^J$. We will prove the theorem for functions $f : \{-1, 1\}^n \to \mathbb{R}$ assuming that it inductively holds for functions $\{-1, 1\} \to \mathbb{R}$ and $\{-1, 1\} \to \mathbb{R}$. There will be four calculations on the way.

First calculation

$$\begin{aligned} \|T_\rho f\|_q &= \left(\mathbb{E}_y\left[\mathbb{E}_x\left[|(T_\rho f)(x,y)|^q\right]\right]\right)^{1/q} \\ &= \mathbb{E}_y\left[\|(T_\rho f)_y(x)\|_q^q\right]^{1/q}. \end{aligned} \tag{4.5}$$

What is $(T_\rho f)_y$ as a function of $x \in \{-1, 1\}^I$.

$$T_\rho f = \sum_{S\subset[n]} \rho^{|S|}\hat{f}(S)u_S = \sum_{A\subset I}\sum_{B\subset J}\rho^{|A|}\rho^{|B|}\hat{f}(A\cup B)u_A u_B,$$

hence as a function of x:

$$(T_\rho f)_y = \sum_{A\subset I}\rho^{|A|}\left(\sum_{B\subset J}\rho^{|B|}\hat{f}(A\cap B)u_B(y)\right)u_A = T_\rho g_y$$

where $g_y : \{-1, 1\} \to \mathbb{R}$ is defined as $g_y = \sum_{A\subset I}\left(\sum_{B\subset J}\rho^{|B|}\hat{f}(A\cap B)u_B(y)\right)u_A$.

Second calculation, where we use the induction assumption:

$$
\begin{aligned}
(4.5) = \mathbb{E}_y\left[\|T_\rho g_y(x)\|_q^q\right]^{1/q} &\leq \mathbb{E}_y\left[\|g_y(x)\|_p^q\right]^{1/q} \\
&= \mathbb{E}_y\left[\mathbb{E}_x\left[|g_y(x)|^p\right]^{q/p}\right]^{1/q} \\
&= \mathbb{E}_y\left[(\text{something non-negative})^{q/p}\right]^{1/q} \\
&= \|\text{something non-negative}\|_{q/p}^{p/q\cdot 1/q}(y) \\
&= \left(\|\mathbb{E}_x\left[|g_y(x)|^p\right]\|_{q/p}(y)\right)^{1/p}. \qquad (4.6)
\end{aligned}
$$

Inside $\|\cdot\|_{q/p}$ in (4.6) is an expectation over x. This expectation is nothing else than constants times a sum over x. Recall the triangle inequality for $\|\cdot\|_{q/p}$, $\|g+h\|_{q/p} \leq \|g\|_{q/p} + \|h\|_{q/p}$. Using the triangle inequality and linearity of the expectation,

$$
\|\mathbb{E}_x[\ldots]\|_{q/p} \leq \mathbb{E}_x\left[\|\ldots\|_{q/p}\right].
$$

So we do the third calculation, where we use the result above:

$$
\begin{aligned}
(4.6) \leq \left(\mathbb{E}_x\left[\|\,|g_y(x)|^p\,\|_q^p\right]\right)^{1/p} &= \left(\mathbb{E}_x\left[\mathbb{E}_y\left[|g_y(x)|^q\right]^{p/q}\right]\right)^{1/p} \\
&= \left(\mathbb{E}_x\left[\|g_y(x)\|_q^p\right](y)\right)^{1/p}. \qquad (4.7)
\end{aligned}
$$

What is $g_y(x)$ as a function of $y \in \{-1,1\}^J$?

$$
\begin{aligned}
g_y(x) &= \sum_{A\subset I}\left(\sum_{B\subset J}\rho^{|B|}\hat{f}(A\cup B)u_B(y)\right)u_A(x) \\
&= \sum_{B\subset J}\rho^{|B|}\left(\sum_{A\subset I}\hat{f}(A\cup B)u_A(x)\right)u_B(y).
\end{aligned}
$$

We see that $g_y(x) = T_\rho h(y)$ for some $h(y)$, i.e. h has the fourier expansion

$$
h = \sum_{B\subset J}\left(\sum_{A\subset I}\hat{f}(A\cup B)u_A(x)\right)u_B = \sum_{S\subset[n]}\hat{f}(S)u_{S\cap I}(x)u_{S\cap J}.
$$

The equation above tells us that h is nothing else than the restriction of f, by fixing x for the coordinates I. We use this and the induction assumption to make the last, fourth, calculation:

$$
\begin{aligned}
(4.7) = \left(\mathbb{E}_x\left[\|T_\rho f_{x\text{ fixed}}\|_q^p\right]\right)^{1/p} &\leq \left(\mathbb{E}_x\left[\|f_{x\text{ fixed}}\|_p^p\right]\right)^{1/p} \\
&= \left(\mathbb{E}_x\left[\mathbb{E}_y\left[|f_{x\text{ fixed}}(y)|^p\right]\right]\right)^{1/p} \\
&= \left(\mathbb{E}\left[|f|^p\right]\right)^{1/p} \\
&= \|f\|_p.
\end{aligned}
$$

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