The behavior of pedestrians near walls in the mean-field approach to crowd dynamics

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Joint work with Boualem Djehiche (KTH)

Pedestrian crowds in confined domains

Example: Unidirectional pedestrian flow

Experimental results show that the average pedestrian speed can be higher in the center of the domain (Daamen et al, 2007) or be higher near the boundary (Zanlungo et al, 2012). Dependent on circumstances (congestion, etc).

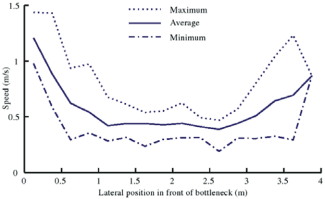

Fig. 5. Speeds as function of the lateral position in a cross-section upstream of the bottleneck during congestion.

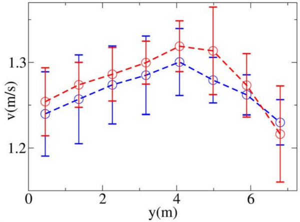

Figure 2. Velocity distributions as measured in the environment E_1 ($\bar{v}^+$ in red, $\bar{v}^-$ in blue). Error bars are obtained as standard deviations of values of $\bar{v}$ averaged over time windows of length 1200 s.

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Treatment of walls in pedestrian crowd models

Modeling approach	**Wall modeling**
Social force	Repulsive forces, disutility
Cellular automata (CA)	Forbidden cells
Continuum limit of CA	Neumann/no-flux boundary conditions
Hughes flow model	Neumann/no-flux boundary conditions, oblique reflection
Mean-field games/control/type games	Neumann/no-flux boundary conditions, disutility

Neumann/no-flux boundary conditions on the pedestrian density correspond to *reflection*.

In this talk we will introduce

sticky reflected SDEs of mean-field type with boundary diffusion

as an alternative way to model walls in the mean-field approach to crowd dynamics.

The mean-field approach to crowd dynamics:

- Interacting system of controlled SDEs.
- Optimal control or differential game setup.
- In the limit (crowd size)$\to \infty$, interaction effects can be written in terms of a mean field (if the interaction is weak, etc.).

Yields: control of McKean-Vlasov equations (Andersson & Djehiche, 2011; Buchdahn et al, 2016; . . .), mean-field games (Lasry&Lions, 2007; Huang et al, 2007; . . .), and mean-field type games (Tembine, 2017).

Consider the SDE system

$$\begin{cases} dX_t = \frac{1}{2} d\ell_t^0(X) + 1_{\{X_t>0\}} dB_t, \quad X_0 = x_0, \\ 1_{\{X_t=0\}} dt = \gamma d\ell_t^0(X), \end{cases} \tag{1}$$

where

- $x_0 \in \mathbb{R}_+$,
- $\gamma \in (0, \infty)$ is a given constant,
- $\ell_0(X)$ is the local time of X at 0,
- B is a standard Brownian motion.

System (1) has no strong solution but a unique weak solution, called a reflected Brownian motion X in $\mathbb{R}_+$ sticky at 0.
See e.g. Engelberg and Peskir (2014).

Grothaus and Vosshall (2017) extended the result to a bounded domain $\mathcal{D} \subset \mathbb{R}^d$ with sticky C^2-smooth boundary $\partial\mathcal{D}$.

Let

- $\Omega := C([0, T]; \mathbb{R}^d)$ be path space,
- $\mathcal{F}$ the Borel σ-field over Ω,
- $X_t(\omega) = \omega(t)$ the coordinate process,

Sticky reflected SDEs of mean-field type with boundary diffusion

To write down the sticky reflected SDE with boundary diffusion system, let

- $n(x)$ be the outward normal of $\partial\mathcal{D}$ at x,
- $\pi(x) := E - n(x)(n(x))^*$, the orthogonal projection on the tangent space of $\partial\mathcal{D}$ at x,
- $\kappa(x) := (\pi(x)\nabla) \cdot n(x)$, the mean curvature of $\partial\mathcal{D}$ at x.

These quantities are uniformly bounded over $\partial\mathcal{D}$.

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These quantities are uniformly bounded over $\partial\mathcal{D}$.

There exists a unique probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ under which

$$\begin{cases} dX_t = 1_{\mathcal{D}}(X_t)dB_t + 1_{\partial\mathcal{D}}(X_t)\left(dB_t^{\partial\mathcal{D}} - \frac{1}{2\gamma}n(X_t)dt\right), \\ dB_t^{\partial\mathcal{D}} = \pi(X_t)\circ dB_t = -\frac{1}{2}\kappa(X_t)n(X_t)dt + \pi(X_t)dB_t, \\ B \text{ standard Brownian motion in } \mathbb{R}^d,\ X_0 = x_0 \in \bar{\mathcal{D}},\ \gamma > 0, \end{cases}$$

and X is $C([0,T];\bar{\mathcal{D}})$-valued $\mathbb{P}$-a.s. (in particular, X is $\mathbb{P}$-a.s. uniformly bounded).

Sticky reflected SDEs of mean-field type with boundary diffusion

$$dX_t = (1_{\mathcal{D}}(X_t) + 1_{\partial\mathcal{D}}(X_t)\pi(X_t))\, dB_t - 1_{\partial\mathcal{D}}(X_t)\frac{1}{2}\left(\kappa(X_t) + \frac{1}{\gamma}\right) n(X_t)dt$$

The sticky reflected SDE with boundary diffusion is composed of špace.1cm

- interior diffusion $1_{\mathcal{D}}(X_t)dB_t$,
- boundary diffusion $1_{\partial\mathcal{D}}(X_t)dB_t^{\partial\mathcal{D}}$
- normal sticky reflection $-1_{\partial\mathcal{D}}(X_t)\frac{1}{2\gamma}n(X_t)dt$

From now on, we abbreviate

$$dX_t =: \sigma(X_t)dB_t + a(X_t)dt.$$

The coefficients σ and a are bounded,

$$\sigma(X_t) := 1_{\mathcal{D}}(X_t) + 1_{\partial\mathcal{D}}(X_t)\pi(X_t),\ a(X_t) := -1_{\partial\mathcal{D}}(X_t)\frac{1}{2}\left(\kappa(X_t) + \frac{1}{\gamma}\right) n(X_t).$$

The stickiness level γ

γ represents the level of stickiness of $\partial\mathcal{D}$.

Let

- λ be the Lebesgue measure on $\mathbb{R}^d$,
- s be the surface measure on $\partial\mathcal{D}$,
- $\rho := 1_{\mathcal{D}}\alpha\lambda + 1_{\partial\mathcal{D}}\alpha' s, \quad \alpha, \alpha' \in \mathbb{R}$.

Choosing

$$\alpha = \bar{\alpha}/\lambda(\mathcal{D}),\ \alpha' = (1-\bar{\alpha})/s(\partial\mathcal{D}),\ \bar{\alpha} \in [0,1],$$

ρ becomes a probability measure on $\mathbb{R}^d$ with full support on $\bar{\mathcal{D}}$.

The measure ρ is in fact the invariant distribution of X_t whenever

$$\frac{1}{\gamma} = \frac{\bar{\alpha}}{(1-\bar{\alpha})}\frac{s(\partial\mathcal{D})}{\lambda(\mathcal{D})}.$$

$\bar{\alpha} \to 1$ as $\gamma \to 0$, and the invariant distribution ρ concentrates on $\mathcal{D}$

$\bar{\alpha} \to 0$ as $\gamma \to \infty$, and the invariant distribution ρ concentrates on $\partial\mathcal{D}$

Interaction and control is introduced via Girsanov transformation.

Let $\mathbb{F}$ be the filtration generated by X completed with the $\mathbb{P}$-null sets of Ω, and

- $|x|_t := \sup_{0 \le s \le t} |x_s|$, $0 \le t \le T$,
- $U \subset \mathbb{R}^d$ and $\mathcal{U} := \{u : [0, T] \times \Omega \to U \mid u\ \mathbb{F}\text{-prog.meas.}\}$,
- $\mathbb{Q}(t) := \mathbb{Q} \circ X_t^{-1}$ denote the t-marginal distribution of X under $\mathbb{Q} \in \mathcal{P}(\Omega)$,
- $\beta : [0, T] \times \Omega \times \mathcal{P}(\mathbb{R}^d) \times U \to \mathbb{R}^d$ such that

(A) $(\beta(t, X, Q(t), u_t))_{t \le T}$ is $\mathbb{F}$-prog.meas. for every $\mathbb{Q} \in \mathcal{P}(\Omega)$ and $u \in \mathcal{U}$.

(B) For every $t \in [0, T]$, $\omega \in \Omega$, $u \in U$, and $\mu \in \mathcal{P}(\mathbb{R}^d)$,

$$|\beta(t, x, \mu, u)| \le C \left(1 + |x|_T + \int_{\mathbb{R}^d} |y| \mu(dy)\right)$$

(C) For every $t \in [0, T]$, $\omega \in \Omega$, $u \in U$, and $\mu, \mu' \in \mathcal{P}(\mathbb{R}^d)$,

$$\left|\beta(t, \omega, \mu, u) - \beta(t, \omega, \mu', u)\right| \le C \cdot d_{TV}(\mu, \mu')$$

Theorem 1

Given $u \in \mathcal{U}$, there exists a unique weak solution ($\mathbb{P}^u$) to the sticky reflected SDE of mean-field type with boundary diffusion

$$dX_t = \sigma(X_t)dB_t^u + \Big(a(X_t) + \sigma(X_t)\beta(t, X_t, \mathbb{P}^u(t), u_t)\Big)dt.$$

Under $\mathbb{P}^u$ the t-marginal distribution of $X_\cdot$ is $\mathbb{P}^u(t)$ for $t \in [0, T]$ and $X_\cdot$ is almost surely $C([0, T]; \bar{\mathcal{D}})$-valued. Furthermore, $\mathbb{P}^u \in \mathcal{P}_p(\Omega)$.

Let

$$f : [0, T] \times \Omega \times \mathcal{P}(\mathbb{R}^d) \times U \to \mathbb{R},$$
$$g : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}.$$

Consider the following finite time-horizon problem:

$$\begin{cases} \min_{u \in \mathcal{U}} \; J(u) = E^u \left[\int_0^T f(t, X, \mathbb{P}^u(t), u_t) dt + g(X_T, \mathbb{P}^u(T)) \right] \\ \\ \\ \end{cases}$$

Weak optimal control of sticky reflected SDEs of mean-field type

Let

$$f : [0, T] \times \Omega \times \mathcal{P}(\mathbb{R}^d) \times U \to \mathbb{R},$$
$$g : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}.$$

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Problem (2) is a weak form mean-field type control problem.
The probability space is controlled via the likelihood L^u.

Additional assumptions on β, f, and g:

(D) For $\phi \in \{\beta, f\}$,

$$\phi_t^u = \phi(t, X, E^u[r_\phi(X_t)], u_t) = \phi(t, X, E[L_t^u r_\phi(X_t)], u_t),$$

and $g_T^u = g(X_T, E[L_T^u r_g(X_T)])$, where $r_\beta, r_f, r_g : \mathbb{R}^d \to \mathbb{R}^d$.

(E) For every $u \in \mathcal{U}$, the process $(f(t, X, E^u[r_f(X_t)], u_t))_t$ is progressively measurable with respect to $\mathbb{F}$ and $(x, y) \mapsto g(x, y)$ is Borel measurable.

(F) The functions $(t, x, y, u) \mapsto (f, \beta)(t, x, y, u)$ and $(x, y) \mapsto g(x, y)$ are twice continuously differentiable with respect to y. Moreover, β, f and g and all their derivatives up to second order with respect to y are continuous in (y, u), and bounded.

In view of (A)-(F) Pontryagin's type stochastic maximum principle is available (Buckdahn et al, 2011, Honsker 2012).

Theorem 2

Assume that $(\hat{u}, L^{\hat{u}})$ is an optimal solution to the mean-field type control problem (2). Then for all $v \in U$ and a.e. $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$\mathcal{H}(L_t^{\hat{u}}, v, p_t, q_t) - \mathcal{H}(L_t^{\hat{u}}, \hat{u}_t, p_t, q_t) + \frac{1}{2}[\delta(L\beta)(t)]^T P_t[\delta(L\beta)(t)] \le 0,$$

where

$$\mathcal{H}(L_t^u, u_t, p_t, q_t) := L_t^u \beta_t^u q_t - L_t^u f_t^u,$$

$$\delta(L\beta)(t) := L_t^{\hat{u}}(\beta(t, X, E[L_t^{\hat{u}} r_\beta(X_t)], v) - \beta_t^{\hat{u}}),$$

$$\begin{cases} dp_t = -\left(q_t \beta_t^{\hat{u}} + E\left[q_t L_t^{\hat{u}} \nabla_y \beta_t^{\hat{u}}\right] r_\beta(X_t) - f_t^{\hat{u}} - E\left[L_t^{\hat{u}} \nabla_y f_t^{\hat{u}}\right] r_f(X_t)\right) dt + q_t dB_t, \\ p_T = -g_T^{\hat{u}} - E\left[L_T^{\hat{u}} \nabla_y g_T^{\hat{u}}\right] r_g(X_T), \\ dP_t = -\Big(\left(\beta_t^{\hat{u}} + E[L_t^{\hat{u}} \nabla_y \beta_t^{\hat{u}}] r_\beta(X_t)\right)^2 P_t + 2\left(\hat{\beta}_t^{\hat{u}} + E[L_t^{\hat{u}} \nabla_y \beta_t^{\hat{u}}] r_\beta(X_t)\right) Q_t \\ \qquad\qquad + E[q_t \nabla_y \beta_t^{\hat{u}}] r_\beta(X_t) - E[\nabla_y f_t^{\hat{u}}] r_f(X_t)\Big) dt + Q_t dB_t, \\ P_T = 0. \end{cases}$$

Identifying optimal controls when U is convex.

Whenever U is convex, the optimality condition simplifies to

$$(v - \hat{u}_t)^* \nabla_u \mathcal{H}(L_t^{\hat{u}}, \hat{u}_t, p_t, q_t) \leq 0, \quad \forall v \in U; \ \mathbb{P}\text{-a.s., a.e.-}t \in [0, T].$$

Assume that $\hat{u}$ is optimal. A matching argument yields

$$q_t = -\nabla_x \phi(X_t, t)\, \sigma(X_t),$$

where $\phi(X_T, T)$ is the terminal condition for p,

$$\phi(X_t, t) := g\left(X_t, E^{\hat{u}}[r_g(X_t)]\right) + E^{\hat{u}}\left[\nabla_y g\left(X_t, E^{\hat{u}}[r_g(X_t))]\right)\right] r_g(X_t),$$

and the optimality condition (variation of $\mathcal{H}$) relates $\hat{u}$ to q,

$$q_t \nabla_u \beta_t^{\hat{u}} = \nabla_u f_t^{\hat{u}}, \quad \mathbb{P}\text{-a.s., a.e.-}t \in [0, T].$$

Example: Unidirectional pedestrian flow

Experimental results show that average pedestrian speed in a cross-section of a corridor can be higher in the center than near the walls (Daamen et al, 2007), but also higher near the walls (Zanlungo et al, 2012), depending on the circumstances (congestion, etc).

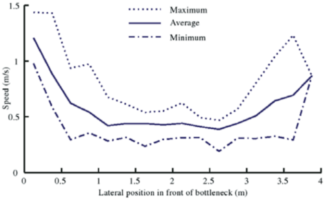

Fig. 5. Speeds as function of the lateral position in a cross-section upstream of the bottleneck during congestion.

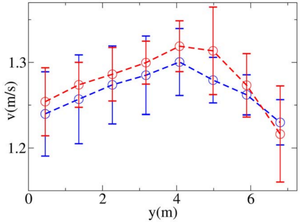

Figure 2. Velocity distributions as measured in the environment E_1 ($\bar{v}^+$ in red, $\bar{v}^-$ in blue). Error bars are obtained as standard deviations of values of $\bar{v}$ averaged over time windows of length 1200 s.
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Example: Unidirectional pedestrian flow

Let $\mathcal{D}$ be a long narrow corridor with exit x_T and entrance x_0 in opposite ends.

$$\begin{cases} \min\limits_{u_\cdot \in \mathcal{U}} \dfrac{1}{2} E\left[\displaystyle\int_0^1 L_t^u f(t, X_\cdot, E[L_t^u r_f(X_t)], u_t)dt + L_T^u |X_T - x_T|^2\right], \\ \text{s.t. } dL_t^u = L_t^u u_t dB_t, \;\; L_0^u = 1. \end{cases}$$

f is a congestion-type running cost:

$$f(t, X_\cdot, E[L_t^u r_f(X_t)], u_t) = \mathcal{C}(X_t)\left\{1 + h\left(t, X_\cdot, E^u[r_f(X_t)]\right)\right\}|u_t|^2,$$

where

- $|u|^2$, $c_f > 0$, is the cost of moving in free space;
- $h|u|^2$ is the additional cost to move in congested areas;
- $\mathcal{C}(X_t) := \xi 1_\Gamma(X_t) + 1_{\mathcal{D}}(X_t)$, $\xi > 0$, monitors f on the boundary $\partial\mathcal{D}$.

Lower ξ yields lower overall cost of moving on $\partial\mathcal{D}$ and vice versa.

Example: Unidirectional pedestrian flow

Assuming U is convex, an optimal control satisfies

$$\hat{u}_t = \frac{\sigma(X_t)(X_t - x_T)}{\mathcal{C}(X_t)\left(1 + h(t, X_\cdot, E^{\hat{u}}[r_f(X_t)])\right)}, \quad \mathbb{P}\text{-a.s., a.e.-}t \in [0, T].$$

$\hat{u}$ implements the following strategy:

- Move towards the exit x_T, but scale the speed according to the local congestion.

$$\hat{u}_t = \frac{\sigma(X_t)(X_t - x_T)}{\mathcal{C}(X_t)\left(1 + h(t, X_\cdot, E^{\hat{u}}[r_f(X_t)])\right)}.$$

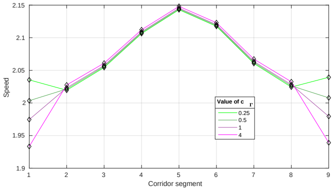
We will compare two congestion costs

- friendly
$$h = h_1 := |X_2(t) - E^{\hat{u}}[X_2(t)]|$$
- averse
$$h = h_2 := \frac{1}{|X_2(t) - E^{\hat{u}}[X_2(t)]|}$$

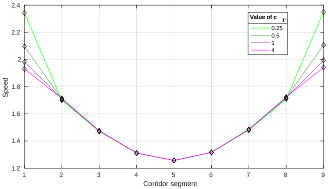
In both cases,

- $r_f((x_1, x_2)) = x_2$
- $X_2(t)$ is the y-component of X_t (perpendicular to the corridor walls).

Estimated cross-section mean speed profiles



(a) Congestion friendly ($h = h_1$).



(b) Congestion averse ($h = h_2$).

- Boundary movement speed is indeed monitored through ξ.

Consider $N \in \mathbb{N}$ (non-transformed, independent) sticky reflected SDEs with boundary diffusion

$$\begin{cases} dX_t^i = a(X_t^i)dt + \sigma(X_t^i)dB_t^i, \\ X_0^i = x_i, \quad i = 1, \dots, N. \end{cases} \tag{3}$$

Grothaus and Vosshall (2017):

There exists a unique probability measure $\mathbb{P}^N$ on $(\mathbf{\Omega}, \mathscr{F})$, where $\mathbf{\Omega} := C([0,T]; \mathbb{R}^{Nd})$ and $\mathscr{F}$ is the corresponding filtration. Under $\mathbb{P}^N$, $(X^1, \dots, X^N)$ satisfies (3) and is $C([0,T]; \bar{\mathcal{D}}^N)$-valued $\mathbb{P}^N$-a.s.

Weak interaction and control can be introduced in the particle system

Given $\mathbf{u} := (u^1, \dots, u^N) \in \mathcal{U}^N$, let $\mu^N(t) := \frac{1}{N} \sum_{i=1} \delta_{X_t^i}$ and

$$dL^{\mathbf{u}}_{i,t} = L^{\mathbf{u}}_{i,t} \beta(t, X^i, \mu^N(t), u^i_t) dB^i_t, \quad L^{\mathbf{u}}_{i,0} = 1, \quad i = 1, \dots, N.$$

$$L^{N,\mathbf{u}}_t := \prod_{i=1}^{N} L^{\mathbf{u}}_{i,t}.$$

$L^{N,\mathbf{u}}_t$ defines a Girsanov transformation of $\mathbb{P}^N$ to $\mathbb{P}^{N,\mathbf{u}}$.

Under $\mathbb{P}^{N,\mathbf{u}}$ the coordinate process is $C([0,T]; \bar{\mathcal{D}})$-valued a.s. and satisfies

$$\begin{cases} dX^i_t = (\sigma(X^i_t)\beta(t, X^i_t, \mu^N(t), u^i_t) + a(X^i_t))dt + \sigma(X^i_t) dB^{i,\mathbf{u}}_t, \\ X^i_0 = x^i_0, \quad i = 1, \dots, N, \end{cases}$$

where $B^{i,\mathbf{u}}$ is a $\mathbb{P}^{N,\mathbf{u}}$-Brownian motion. Also, $\mathbb{P}^{N,\mathbf{u}} \in \mathcal{P}_p((C([0,T]; \bar{\mathcal{D}})^N)$.

Social cost for the particle system:

$$J_N(\mathbf{u}) := \frac{1}{N}\sum_{i=1}^{N} E^{N,\mathbf{u}}\left[\int_0^T f(t, X^i, \mu^N(t), u_t^i)dt + g(X_T^i, \mu^N(T))\right]$$

Minimization of $J_N(\mathbf{u})$ is a cooperative scenario.

If the mean-field optimal control is closed-loop, the mean-field system can be approximated by a particle system and the mean-field cost by a social cost. The theorem on the next page relies on Theorem 3.2 of Lacker (2018).

Theorem 3

Let $u \in \mathcal{U}$ be a closed-loop control, i.e. $u_t(\omega) = \varphi(\omega_{\cdot\wedge t})$ for some measurable function $\varphi : (\Omega, \mathcal{F}) \to (U, \mathcal{B}(U))$. Given the control u and a random variable ξ with law λ (nonatomic with support only on $\bar{\mathcal{D}}$), the sticky reflected SDE of mean-field type with boundary diffusion

$$\begin{cases} dX_t = (a(X_t) + \sigma(X_t)\beta(t, X_\cdot, \mathbb{P}^u(t), \varphi(X_{\cdot\wedge t})))\, dt + \sigma(X_t)dB_t, \\ X_0 = \xi, \end{cases} \tag{4}$$

can be approximated by the interacting particle system with all components using the fixed closed-loop control u. Furthermore, the value of the mean-field cost functional J at u is the asymptotic social cost of the interacting particle system as $N \to \infty$ when all the $X^{N,i}$s are using the fixed control u. More specifically,

$$\lim_{N\to\infty} D_T\left(\mathbb{P}^{N,\mathbf{u}} \circ (X_\cdot^{N,1}, \ldots, X_\cdot^{N,k})^{-1}, (\mathbb{P}^u \circ X_\cdot^{-1})^{\otimes k}\right) = 0, \tag{5}$$

with $\mathbf{u} = (u, \ldots, u)$, and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} J^i(u, \ldots, u) \to J(u). \tag{6}$$

Conclusions

- Mean-field approach to crowd dynamics
  - congestion, crowd aversion, etc.
  - decision-based modeling with anticipating agents
  - correspondence between micro- and macroscopic picture

- Sticky reflected SDEs of mean-field type with boundary diffusion
  - as an alternative to reflective boundary conditions in confined domains
  - pedestrians no longer "bounce" at the boundary
  - pedestrians may interact and take actions while spending time at the boundary
  - corresponds to a microscopic model

Thank you!

Assume that $(\hat{u}, \hat{L})$ is an optimal solution for the mean-field type control problem. Recall the first order adjoint equation,

$$\begin{cases} dp_t = -\Big(q_t \beta_t^{\hat{u}} + E\left[q_t L_t^{\hat{u}} \nabla_y \beta_t^{\hat{u}} \right] r\beta(X_t) \\ \qquad\qquad - f_t^{\hat{u}} - E\left[L_t^{\hat{u}} \nabla_y f_t^{\hat{u}} \right] r_f(X_t) \Big) dt + q_t dB_t, \\ p_T = -g_T^{\hat{u}} - E\left[L_T^{\hat{u}} \nabla_y g_T^{\hat{u}} \right] r_g(X_T). \end{cases} \tag{7}$$

Rewriting $E[L_t^{\hat{u}} Y_t] = E^{\hat{u}}[Y_t]$ and changing measure to $\mathbb{P}^{\hat{u}}$,

$$\begin{cases} dp_t = -\Big(E^{\hat{u}}\left[q_t \nabla_y \beta_t^{\hat{u}} \right] r\beta(X_t) - f_t^{\hat{u}} - E^{\hat{u}}\left[\nabla_y f_t^{\hat{u}} \right] r_f(X_t) \Big) dt + q_t dB_t^{\hat{u}}, \\ p_T = -g_T^{\hat{u}} - E^{\hat{u}}\left[\nabla_y g_T^{\hat{u}} \right] r_g(X_T). \end{cases}$$

Examples: Convex and compact U

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Rewriting $E[L_t^{\hat{u}} Y_t] = E^{\hat{u}}[Y_t]$ and changing measure to $\mathbb{P}^{\hat{u}}$,

$$\begin{cases} dp_t = -\Big(E^{\hat{u}}\left[q_t \nabla_y \beta_t^{\hat{u}} \right] r\beta(X_t) - f_t^{\hat{u}} - E^{\hat{u}}\left[\nabla_y f_t^{\hat{u}} \right] r_f(X_t) \Big) dt + q_t dB_t^{\hat{u}}, \\ p_T = -g_T^{\hat{u}} - E^{\hat{u}}\left[\nabla_y g_T^{\hat{u}} \right] r_g(X_T). \end{cases}$$

Whenever U is convex, the optimality condition simplifies to

$$\mathcal{H}(\hat{L}_t, v, p_t, q_t) - \mathcal{H}(\hat{L}_t, \hat{u}_t, p_t, q_t) \leq 0, \quad \forall v \in U;\ \mathbb{P}\text{-a.s., a.e.-}t \in [0, T].$$

Example: Convex and compact U

p part of the solution to a BSDE so it is the conditional expectation

$$p_t = -E^{\hat{u}}\left[\phi\left(X_T, T\right) \mid \mathcal{F}_t\right] + E^{\hat{u}}\left[\int_t^T (\dots) ds \mid \mathcal{F}_t\right], \tag{8}$$

where as before

$$\phi(X_t, t) := g\left(X_t, E^{\hat{u}}[r_g(X_t)]\right) + E^{\hat{u}}\left[\nabla_y g\left(X_t, E^{\hat{u}}[r_g(X_t))]\right)\right] r_g(X_t).$$

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By Dynkin's formula,

$$E^{\hat{u}}[\phi(X_T, T) \mid \mathcal{F}_t] = \phi\left(X_t, t\right) + \int_t^T E^{\hat{u}}[(\dots)(s) \mid \mathcal{F}_t] ds.$$

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The optimality condition (variation of $\mathcal{H}$) relates $\hat{u}$ to q,

$$q_t \nabla_u \beta_t^{\hat{u}} = \nabla_u f_t^{\hat{u}}, \quad \mathbb{P}\text{-a.s., a.e.-}t \in [0, T].$$

Consider on some admissible domain $\mathcal{D} \subset \mathbb{R}^d$ the mean-field LQ problem of minimizing final variance

$$\begin{cases} \min_{u \in \mathcal{U}} \frac{1}{2} E\left[\int_0^T L_t^u |u_t|^2 dt + L_T^u \left|X_T - E[L_T^u X_T]\right|^2\right], \\ \text{s.t. } dL_t^u = L_t^u u_t dB_t, \quad L_0^u = 1, \end{cases}$$

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$\hat{u}$ takes $\mathbb{P}$ to $\mathbb{P}^{\hat{u}}$ under which the coordinate process solves the non-linear SDE

$$dX_t = \left(a(X_t) - \sigma(X_t)(X_t - E^{\hat{u}}[X_t])\right) dt + \sigma(X_t) dB_t^{\hat{u}}.$$

Total variation distance on $\mathcal{P}(\Omega)$

For $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, the total variation distance is defined by the formula

$$d(\mu, \nu) = 2 \sup_{B \in \mathcal{B}(\mathbb{R}^d)} |\mu(B) - \nu(B)|. \tag{9}$$

Define on $\mathcal{F}$ the total variation metric

$$d(P, Q) := 2 \sup_{A \in \mathcal{F}} |P(A) - Q(A)|. \tag{10}$$

On the filtration $\mathbb{F}$,

$$D_t(Q, \widetilde{Q}) := 2 \sup_{A \in \mathcal{F}_t} |Q(A) - \widetilde{Q}(A)|, \quad 0 \leq t \leq T. \tag{11}$$

It satisfies

$$D_s(Q, \widetilde{Q}) \leq D_t(Q, \widetilde{Q}), \quad 0 \leq s \leq t. \tag{12}$$

For $Q, \widetilde{Q} \in \mathcal{P}(\Omega)$ with time marginals $Q_t := Q \circ x_t^{-1}$ and $\widetilde{Q}_t := \widetilde{Q} \circ x_t^{-1}$, then

$$d(Q_t, \widetilde{Q}_t) \leq D_t(Q, \widetilde{Q}), \quad 0 \leq t \leq T. \tag{13}$$

Endowed with the total variation metric D_T, $\mathcal{P}(\Omega)$ is a complete metric space. Moreover, D_T carries out the usual topology of weak convergence.