Relaxed optimal control

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Example 1

Let $U = \{-1, 1\}$ be the set of control values.

Let $\mathcal{U}[0, 1]$ be the set of all measurable functions

$$u : [0, 1] \to U.$$

An element of $\mathcal{U}[0, 1]$ is called an **admissible control**.

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$$x^u(t) = \int_0^t u(s)ds, \quad t \in [0,1], \; u \in \mathcal{U}[0,1].$$

We want to minimize the cost functional

$$J(u) = \int_0^1 x^u(s)^2 ds$$

over $\mathcal{U}[0,1]$.

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A sequence $(u_n)_n$ such that $J(u_n) \to 0$ can be constructed. Let

$$u_n(t) = (-1)^k, \quad \text{if } t \in \left[\frac{k}{n}, \frac{(k+1)}{n}\right), \; 0 \leq k \leq n-1$$

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Then $|x^{u_n}(t)| \leq n^{-1}$, which implies $J(u_n) \leq n^{-2}$. Therefore

$$\inf_{u \in \mathcal{U}[0,1]} J(u) = 0.$$

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There is no $u \in \mathcal{U}[0,1]$ such that $J(u) = 0$!

$J(u) = 0 \Rightarrow x^u(t) = 0 \ \forall t \in [0,1]$. This in turn implies that $u(t) = 0$ which is not in $\mathcal{U}[0,1]$.

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Relaxed controls allows us to find a limit in a larger space. Each $u \in \mathcal{U}[0,1]$ with the $\mathcal{P}(U)$-valued process $(\delta_{u(t)};\ t \in [0,1])$ through the map

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Define $q_n(dt, da) := \delta_{u_n(t)}(da)dt \in \mathcal{P}([0,1] \times U)$ for previously defined u_n. Does $q_n(dt, da)$ converge?

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Claim 2

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For any $\varphi \in C_b([0,1] \times U)$,

$$\int_{[0,1]\times U} \varphi(t,a) q_n(dt, da) = \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \varphi(t, (-1)^k) dt$$

Since $[0,1]$ is compact, $t \mapsto \varphi(t, \pm 1)$ is uniformly continuous over $[0,1]$. So given $\varepsilon > 0$, there exists an $m_0 > 0$ such that for all $m \geq m_0$, $|\varphi(t,a) - \varphi(s,a)| < \varepsilon$ whenever $|t - s| < m^{-1}$.

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Fix $m > m_0$ and let $n = 2m$. We have

$$\int_0^1 \varphi(t,a)dt = \sum_{j=0}^{m-1} \int_{\frac{2j}{2m}}^{\frac{2j+1}{2m}} \varphi(t,a)dt + \int_{\frac{2j+1}{2m}}^{\frac{2j+2}{2m}} \varphi(t,a)dt.$$

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Claim 2 cont.

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For each $j \in \{0, \ldots, m-1\}$, the Mean-Value Theorem yields

$$\left| \int_{\frac{2j}{2m}}^{\frac{2j+1}{2m}} \varphi(t, a)dt - \int_{\frac{2j+1}{2m}}^{\frac{2j+2}{2m}} \varphi(t, a)dt \right| < \frac{\varepsilon}{2m}$$

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Hence, for $n = 2m$, we have

$$\left| \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \varphi(t, (-1)^k)dt - \frac{1}{2} \int_0^1 \varphi(t, -1) + \varphi(t, 1)dt \right| < \frac{\varepsilon}{2}$$

The case $n = 2m + 1$ is treated in similar fashion.

Example 1

Consider the control problem associated with $\mathcal{P}(U)$-valued processes $\mu = (\mu_t;\ t \in [0,1])$,

$$\begin{aligned} \text{minimize} \quad & \mathcal{J}(\mu) = \int_0^1 (x^\mu(t))^2 dt \\ \text{subject to} \quad & x^\mu(t) = \int_0^t \int_U a\mu_s(da)ds. \end{aligned}$$

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Again $\inf_\mu \mathcal{J}(\mu) = 0$. For $\mu^*(da)dt := \frac{1}{2}(\delta_{-1} + \delta_1)(da)dt$ we have $x^{\mu^*}(t) = 0,\ t \geq 0$, which implies that $\mathcal{J}(\mu^*) = 0$. Hence

$$\inf_\mu \mathcal{J}(\mu) = \mathcal{J}(\mu^*).$$

Example 1

Moreover,

$$\inf_u J(u) = \inf_\mu \mathcal{J}(\mu)$$

A candidate for the set of relaxed controls is $\mathcal{R} \subset \mathcal{P}([0,1] \times U)$ such that

- $q(da, dt)$ projected on U coincides with a ($\mathcal{F}_t$-adapted) $\mathcal{P}(U)$-valued process $\mu_t(da)$,
- $q(da, dt)$ projected on $[0,1]$ coincides with the Lebesgue measure dt.

Essentially: $q(da, dt) = \mu_t(da)dt$.

Set of relaxed controls

Let (U, d) be a separable metric space. Example suggests that the set of admissible controls $\mathcal{U}[0, T]$ embeds into $\mathcal{R}$ through the map

$$\Psi : u \in \mathcal{U}[0, T] \mapsto \Psi(u)(dt, da) = \delta_{u(t)}(da)dt \in \mathcal{R}$$

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Strict control: at each t we assign a fixed value $u(t) \in U$ to the control process.
Relaxed control : at each t we randomly choose a control from U with (random) probability $\mu_t(da)$.

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In Example 1: $\inf_{\mu \in \mathcal{R}} \mathcal{J}(\mu) = \inf_{u \in \mathcal{U}[0,1]} J(u)$. When can we expect this?

The full stochastic control problem

Let U, $\mathcal{U}[0,T]$ and $\mathcal{R}$ be defined in line with previous slides. Let

$$dx(t) = b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW_t,$$
$$x(0) = x_0.$$

We want to minimize

$$J(u) = \mathbb{E}\left[\int_0^T f(t, x(t), u(t))dt + h(X(T))\right], \ u \in \mathcal{U}[0,T].$$

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The relaxed cost functional is

$$\mathcal{J}(\mu) = \mathbb{E}\left[\int_0^T \int_U f(t, x(t), a)\mu_t(da)dt + h(x(T))\right], \; \mu \in \mathcal{R}.$$

Standing assumption: b, σ, f, h are bounded and continuous in (x, u).

Strong vs weak solutions of the dynamics

We can solve the dynamics in a strong (pathwise) or a weak (distributional) sense.

Strong solution:
Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \in [0, T]), \mathbb{P})$, an $\mathcal{F}_t$-adapted standard Wiener process W, an admissible control $u \in \mathcal{U}[0,1]$ and an initial value x_0, an $\mathcal{F}_t$-adapted continuous process $(x(t);\ t \in [0,1])$ is a strong solution if

$$x(t) = x_0 + \int_0^t b(s, x(s), u(s))ds + \int_0^t \sigma(s, x(s), u(s))dW_s,\ \mathbb{P}-\text{a.s.}$$

together with some integrability of the coefficients.

Strong vs weak solution of the dynamics

We can solve the dynamics in a strong (pathwise) or a weak (distributional) sense.

Weak control:
The tuple $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, u, x)$ is called a weak control if

- $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a filtered probability space
- u is a $\mathcal{F}_t$-adapted U-valued process.
- x is and $\mathcal{F}_t$-adapted and continuous process such that $x(0) = x_0$ and

$$M^\varphi(t) := \varphi(x(t)) - \varphi(x(0)) - \int_0^t L_s^u \varphi(x(s))ds$$

  is a $\mathbb{P}$-martingale for each $\varphi \in C_b^2(\mathbb{R})$.

Here, L^u is infinitesimal generator associated to the the dynamics

$$L_t^u \varphi(x) = \frac{1}{2}\sigma^2(t, x, u)\varphi''(x) + b(t, x, u)\varphi'(x).$$

Strong vs weak relaxation of the dynamics

The two types of solution suggest two types of relaxation.

Strong relaxation:
Integrate the coefficients b and σ against the relaxed control $\mu_t(da)$,

$$\begin{aligned} x(t) = x_0 &+ \int_0^t \int_U b(s, x(s), a)\mu_s(da)ds \\ &+ \int_0^t \int_U \sigma(s, x(s), a)\mu_s(da)dW_s \end{aligned}$$

Strong vs weak relaxation of the dynamics

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Weak relaxed control:
The tuple $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, \mu, x)$ is called a weak control if

- $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a filtered probability space
- μ is a $\mathcal{F}_t$-adapted $\mathcal{P}(U)$-valued process such that $\mathbb{I}_{(0,t]}\mu_t$ is $\mathcal{F}_t$-measurable.
- x is and $\mathcal{F}_t$-adapted and continuous process such that $x(0) = x_0$ and

$$M^\varphi(t) := \varphi(x(t)) - \varphi(x(0)) - \int_0^t \int_U L_s^a \varphi(x(s))\mu_s(da)ds$$

  is a $\mathbb{P}$-martingale for each $\varphi \in C_b^2(\mathbb{R})$.

Here, L^u is infinitesimal generator associated to the the dynamics

Young measure

Theorem 1

Assume that the sequence $(u_n)_n$ of $\mathcal{F}_t$-predictable and U-valued controls is uniformly integrable,

$$\lim_{c\to\infty}\sup_n \mathbb{E}\left[\int_0^T |u_n(t)|\mathbb{I}_{\{|u_n(t)|\geq c\}}dt\right] = 0.$$

Then there exists a subsequence $(u_{n_j})_j$ of $(u_n)_n$ and, for a.e. $t \in [0, T]$, a random probability measure μ_t on U such that

$$\delta_{u_{n_j}(t)}(da)dt \text{ converges weakly to } \mu_t(da)dt,\ \mathbb{P}-\text{a.s.}$$

The process $(\mu_t(da);\ t \in [0, T])$ is called the family of Young measures associated with the subsequence $(u_{n_j})_j$.

Young measure

A more restricted situation:

Lemma 1

Assume that U is a convex and compact subset of $\mathbb{R}^d$. Then there for all relaxed controls $\mu_t(da)dt$ there exists a strict control u such that

$$\int_0^t \int_U a\mu_s(da)ds = \int_0^t u(s)ds, \quad t \in [0, T], \ \mathbb{P} - \text{a.s.}$$

Chattering Lemma

Young measure: get relaxed control from sequence of strict controls.

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Theorem 2

Assume that U is a compact set. Let (μ_t) be a predictable $\mathcal{P}(U)$-valued process. Then there exists a sequence $(u_n(t))_n$ of predictable U-valued processes such that

$$\delta_{u_n(t)}(da)dt \Rightarrow \mu_t(da)dt, \ \mathbb{P} - \text{a.s.}$$

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Can it be so that with Chattering Lemma and some continuity of $\mathcal{J}$, we have $\inf_{\mu \in \mathcal{R}} \mathcal{J}(\mu) \geq \inf_{u \in \mathcal{U}[0,T]} J(u)$?

Example 2

Let $U = \{-1, 1\}$ and consider the following problem

$$\begin{aligned} \text{minimize} \quad & J(u) = \mathbb{E}\left[h(x(1))\right] \\ \text{subject to} \quad & x(t) = x_0 + \int_0^t u(s) dW_s. \end{aligned}$$

where h is some smooth function. Since $u \in \{-1, 1\}$, $\langle x \rangle_t = t$ and $x(t) - x_0$ is a standard Wiener process. Therefore

$$g(t, x_0) = \inf_{u \in \mathcal{U}[0,1]} \mathbb{E}\left[h(x_0 + \int_0^t u(s) dW_s)\right]$$

satisfies the heat equation

$$\frac{\partial g}{\partial t}(t, x) = \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, x), \quad g(0, x) = h(x).$$

Example 2

The heat equation implies that $g(t,x) \neq h(x), t > 0$. Consider the relaxed control $\mu_t(da) = \frac{1}{2}(\delta_{-1}(da) + \delta_1(da))$. The strongly relaxed control is

$$x(t) = x_0 + \int_0^1 \int_U a\mu_s(da)dW_s = x_0 + \int_0^1 \frac{1}{2}(-1+1)dW_s = x_0,$$

So

$$\mathcal{J}(\mu) = \mathbb{E}\left[h(x_0 + \int_0^1 \int_U a\mu_s(da)dW_s)\right] = \mathbb{E}\left[h(x_0)\right] = h(x_0)$$

and

Example 3: $U = \{a_1, \ldots, a_n\}$

Every relaxed control $\mu_t(da)dt$ is a convex combination of Dirac measures on the elements of U,

$$\mu_t(da)dt = \sum_{i=1}^{n} c_t^i \delta_{a_i}(da)dt, \tag{1}$$

c_t^i is a $[0,1]$-valued process and $\sum_{i=1}^{n} c_t^i = 1$.

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$$M^\varphi(t) = \varphi(x(t)) - \varphi(x(0)) - \int_0^t \underbrace{\sum_{i=1}^{n} c_s^i L_s^{a_i}}_{=:\mathcal{L}_s} \varphi(x(s))ds. \tag{2}$$

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Note that $M^\varphi(t) = \int_0^t d\varphi(x(s)) - \int_0^t \mathcal{L}_s\varphi(x(s))ds$ where

$$\mathcal{L}_s\varphi(x(s))ds = \sum_{i=1}^{n} c_s^i b(s, x(s), a_i)\varphi'(x(s))ds + \sum_{i=1}^{n} c_s^i \frac{1}{2}\sigma\sigma^*(s, x(s), a_i)\varphi''(x(s))ds \qquad (3)$$

$$d\varphi(x(s)) = \varphi'(x(s))dx(s) + \frac{1}{2}\varphi''(x(s))d\langle x\rangle_s$$

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&\quad + \sum_{i=1}^{n} c_s^i \frac{1}{2}\sigma\sigma^*(s, x(s), a_i)\varphi''(x(s))ds \qquad (3) \\
d\varphi(x(s)) &= \varphi'(x(s))dx(s) + \frac{1}{2}\varphi''(x(s))d\langle x\rangle_s
\end{aligned}$$

For the strong relaxation,

$$\begin{aligned}
dx(s) &= \sum_{i=1}^{n} c_s^i b(s, x(s), a_i)ds + \sum_{i=1}^{n} c_s^i \sigma(s, x(s), a_i)dW_s \\
d\langle x\rangle_s &= \left(\sum_{i=1}^{n} c_s^i \sigma(s, x(s), a_i)\right)^2 ds
\end{aligned} \qquad (4)$$

A characterization of the weakly relaxed process

Def: Orthogonal martingale measure

The random function $m : \Omega \times [0,T] \times U$ is a continuous martingale measure with covariance measure $\nu : [0,T] \times U \times U$ if

- $m(\cdot, A)$ is a continuous square-integrable martingale for all $A \in \mathcal{B}(U)$,
- the process

$$m(t,A)m(t,B) - \int_{[0,t]\times A\times B} \nu(dt,dx,dy) \qquad (5)$$

is a martingale. If ν is supported on the diagonal of the set $U \times U$, i.e. $\nu(dt,dx,dy) = \delta_x(dy)\widetilde{\nu}(dx,dt)$, then m is an orthogonal martingale measure with intensity $\widetilde{\nu}$.

A characterization of the weakly relaxed process

Theorem 3

Let $\mathbb{P}$ be the solution to relaxed martingale problem. Then $\mathbb{P}$ is the probability law of x satisfying

$$dx(t) = \int_U b(t, x(t), a)\mu_t(da)dt + \int_U \sigma(t, x(t), a)m(dt, da) \tag{6}$$

where m is an orthogonal continuous martingale measure with intensity $\mu_t(da)dt$.

A characterization of the weakly relaxed process

Theorem 4

Let m be a continous orthogonal martingale-measure with intensity $\mu_t(da)dt$. Then there exists a Wiener process W and a sequence of predictable U-valued processes (u_n) such that for all continuous and bounded $\varphi : U \to \mathbb{R}$ and for all $t \in [0, T]$

$$\lim_{n\to\infty} \mathbb{E}\left[\left(m_t(\varphi) - \int_0^t \varphi(u_n(s))dW_s\right)^2\right] = 0 \qquad (7)$$

where $m_t(\varphi) = \int_0^t \int_U \varphi(a)m(ds, da)$.

A characterization of the weakly relaxed process

For the strongly relaxed dynamics, the martingale measure is

$$\begin{aligned} m(t,A) &= \int_0^t \int_A \mu_s(da)dW_s \\ &= \int_0^t \int_A \sum_{i=1}^n c_s^i \delta_{a_i}(da)dW_s = \int_0^t \sum_{i=1}^n c_s^i \mathbb{I}_{\{a_i \in A\}} dW_s \end{aligned} \tag{8}$$

The quadratic variation process is not supported only on the diagonal of $U \times U$!

$$\nu(dt, da, db) = \mu_t(da)\mu_t(db)dt \tag{9}$$

Example 2: $U = \{a_1, \dots, a_n\}$

Candidate orthogonal martingale measure:

$$m(t, A) = \int_0^t \sum_{i=1}^n \sqrt{c_s^i} \mathbb{I}_{a_i \in A} dW_s^i \tag{10}$$

Indeed,

$$\nu(dt, da, db) = \delta_a(db) \underbrace{\sum_{i=1}^n \sqrt{c_s^i} \delta_{a_i}(da) dt}_{= \mu_t(da) dt} \tag{11}$$

Thus the weakly relaxed dynamics are

$$\begin{aligned} dx(t) &= \int_U b(t, x(t), a) \mu_t(da) dt + \int_U \sigma(t, x(t), a) m(dt, da) \\ &= \sum_{i=1}^n b(t, x(t), a_i) c_t^i dt + \sum_{i=1}^n \sigma(t, x(t), a_i) \sqrt{c_t^i} dW_t^i \end{aligned} \tag{12}$$

Conclusions

Summary:

- $\inf_{u \in U} J(u) = \inf_{\mu \in \mathcal{R}} \mathcal{J}(\mu)$
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Further applications of relaxed control

- Decision theory (posterior risk)
- Game theory (mixed strategies)