Stochastic Dynamic Graphon Games

The Linear-Quadratic Case

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Introduction to Mean Field Games and Applications
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Talk based on the joint work

"Stochastic Graphon Games: II. The Linear-Quadratic Case" A., Carmona, Laurière, arXiv 2021.

→ The graphon game: A limit model for a class of linear-quadratic stochastic games with non-identical players

→ Convergence analysis

"Finite State Graphon Games with Applications to Epidemics" A., Carmona, Dayanıklı, Laurière, arXiv 2021.

Introductory example

Introductory example: MFG for systemic risk

N-player game with weak interaction (Carmona, Fouque, Sun '13)

$$\min_{\alpha^k} \mathbb{E}\left[\int_0^T \frac{1}{2}(\alpha_t^k)^2 - q\alpha_t^k \frac{1}{N}\sum_{j=1}^N (X_t^j - X_t^k) + \frac{\varepsilon}{2}\left(\frac{1}{N}\sum_{j=1}^N (X_t^j - X_t^k)\right)^2 dt + g(X_T^k)\right]$$

$$dX_t^k = a\left(\frac{1}{N}\sum_{j=1}^N (X_t^j - X_t^k) + \alpha_t^k\right) dt + \sigma dW_t^k, \quad k = 1, \ldots, N$$

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The interaction term is the average log-monetary reserve difference:

$$\frac{1}{N}\sum_{i=1}^N (X_t^j - X_t^i).$$

"... representing the rate at which bank i borrows from or lends to bank j."

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What if there is a distinguishing feature, impacting the interaction, such as ...?

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A very simple geography:

→ each bank is **geographically labeled**, bank k by $x_k \in I := [0,1]$

→ **weights** $w : I \times I \mapsto [0,1]$, symmetric

- x_k and x_j geographical close $\Rightarrow w(x_k, x_j)$ is large
- x_k and x_j distant $\Rightarrow w(x_k, x_j)$ is small

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Instead of $\frac{1}{N}\sum_{j=1}^{N}(X_t^j - X_t^k)$, player k now "feels" **the weighted aggregate**

$$Z_t^{k,N} := \frac{1}{N}\sum_{j=1}^{N} w(x_k, x_j)(X_t^j - X_t^k)$$

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N-player game for systemic risk with **non-identical players**

$$\min_{\alpha^k} \mathbb{E}\left[\int_0^T \frac{1}{2}(\alpha_t^k)^2 - q\alpha_t^k Z_t^{k,N} + \frac{\varepsilon}{2}\left(Z_t^{k,N}\right)^2 dt + g(X_T^k)\right]$$

$$dX_t^k = a\left(Z_t^{k,N} + \alpha_t^k\right) dt + \sigma dW_t^k$$

Introductory example: MFG for systemic risk

Limit model with infinitesimal agents

Drawing inspiration from economic theory:

- $\rightarrow$ If the agent (index) space is an **atomless probability space** $(I, \mathcal{I}, \lambda)$ then each individual agent has no influence.
- $\rightarrow$ Aggregates are averages over the **agent space** (cf. expectations)
- $\rightarrow$ Agents are exposed to **idiosyncratic shocks** (independent noise)

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A candidate continuum limit model

$$\min_{\alpha^x} \mathbb{E}\left[\int_0^T \frac{1}{2}(\alpha_t^x)^2 - q\alpha_t^x Z_t^x + \frac{\varepsilon}{2}(Z_t^x)^2\, dt + g(X_T^x)\right]$$

$$dX_t^x = a\,(Z_t^x + \alpha_t^x)\,dt + \sigma dW_t^x, \quad t \in [0,T],\ x \in I$$

$$Z_t^x = \int_I w(x,y)(X_t^y - X_t^x)dy, \quad t \in [0,T],\ x \in I$$

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$$Z_t^x = \int_I w(x, y)(X_t^y - X_t^x)dy, \quad t \in [0, T],\ x \in I$$

(Q1) **Is the (linear-quadratic) limit model well defined?**

(Q2) **Can we compute and approximate Nash equilibria?**

Introductory example: MFG for systemic risk

The candidate limit model is building on the Brownian motion vector $(B^x)_{x \in I}$.

Measurability problems when λ is the Lebesgue measure

→ For an iid Brownian motion $(B^x)_{x \in I}$ based on the usual continuum product via Kolmogorov construction: **almost all sample functions $x \mapsto B^x(\omega)$ are essentially equal to an *arbitrarily* given function $x \mapsto \beta^x$ on $[0, 1]$.**

→ A process like that is not measurable in the index – problem defining aggregates!

Relating back to the motivating example:

→ Interaction term is the aggregate $\int_I w(x, y)(X_t^y - X_t^x)dy$

→ $y \mapsto X_t^y$ is *a priori* **not *dy*-measurable!**

Outline

"Stochastic Graphon Games: II. The Linear-Quadratic Case" A., Carmona, Lauriére, arXiv 2021.

→ The graphon game: A limit model for linear-quadratic stochastic games with non-identical players

- Measurability problems addressed with Fubini extension theory
- (Linear) Graphon SDE
- Nash equilibria for LQ Graphon games

→ Convergence analysis

"Finite State Graphon Games with Applications to Epidemics" A., Carmona, Dayanıklı, Lauriére, arXiv 2021.

→ Finite state graphon games

- Pure-jump Graphon SDE
- Applications to epidemiology

The Linear Graphon SDE System

The Linear Graphon SDE system

We will consider the following "linear SDE" system: given an admissible strategy profile $\underline{\alpha} \in \underline{\mathcal{A}}$ (open-loop; decentralized; progressive; square integrable)

$$\begin{cases} dX_t^x = \Big(a(x)X_t^x + b(x)\alpha_t^x + c(x)Z_t^x \Big) dt + dB_t^x, & t \in [0, T],\ x \in I \\ Z_t^x = \int_I w(x,y) X_t^y \lambda(dy), & t \in [0, T],\ x \in I, \\ X_0^x = \xi^x \end{cases}$$

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The graphon w

→ is a symmetric measurable function from $I \times I$ to $[0,1]$

→ induces a Hilbert-Schmidt operator $W : L^2(I) \to L^2(I)$

→ W can be extended to $L^2_\lambda(I)$ (and beyond)

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(Q1) **Is the (linear-quadratic) limit model well defined?**

Is there a suitable probability space over $\Omega \times I$ where we can

→ define the idiosyncratic noise $(B^x)_{x \in I}$ so that

→ the aggregates $(Z_t^x)_{x \in I}$ are well-defined and

→ in which we can solve the system (1) is a strong sense?

Fubini extension

Theory developed by (Sun '98, '06; Sun, Zhang '09) and others.

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1. is atomless
2. extends the Lebesgue space $(I, \mathcal{B}(I), \text{Leb}(I))$

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→ carrying an **essentially pairwise independent** Brownian motion vector $(B^x)_{x \in I}$

for λ-a.e. $x \in I$ B^x is independent of B^y for λ-a.e. $y \in I$

The Linear Graphon SDE System

Fubini extension

To emphasize the Fubini property ($\mathbb{Q}$ disintegrates with marginals $\mathbb{P}$ and λ) we write

$$(\Omega \times I, \mathcal{F} \boxtimes \mathcal{I}, \mathbb{P} \boxtimes \lambda) := (\Omega \times I, \mathcal{W}, \mathbb{Q})$$

We will denote L^2-spaces over the Fubini extension by $L^2_{\boxtimes}$, expectation by $\mathbb{E}^{\boxtimes}$.

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Theorem (Exact Law of Large Numbers)

Let f be a process from $(\Omega \times I, \mathcal{F} \boxtimes \mathcal{I}, \mathbb{P} \boxtimes \lambda)$ to the Polish space S. If $(f^x)_{x \in I}$ are e.p.i., then $\lambda \circ [f^\cdot(\omega)]^{-1} = (\mathbb{P} \boxtimes \lambda) \circ [f^\cdot(\cdot)]^{-1}$, $\mathbb{P}$-a.s.

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Corollary: if f is furthermore $\mathbb{P} \boxtimes \lambda$-integrable:

$$\int_A f^x(\omega)\lambda(dx) = \int_A \mathbb{E}[f^x]\lambda(dx), \quad A \in \mathcal{I}, \ \mathbb{P}\text{-a.s.}$$

"Complete removal of individual uncertainty", "Insurable risk"

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"Complete removal of individual uncertainty", "Insurable risk"

Example

Since $(B^x_t)_{x\in I}$ are e.p.i. and integrable

$$\int_A B^x_t(\omega)\lambda(dx) = \int_A \mathbb{E}[B^x_t]\lambda(dx) = 0, \quad \mathbb{P}\text{-a.s.}, \quad A \in \mathcal{I},\ t \in [0, T].$$

The Linear Graphon SDE System

With $(B^x)_{x\in I}$ and λ as introduced in the discussion above

$$\begin{cases} dX_t^x = \Big(a(x)X_t^x + b(x)\alpha_t^x + c(x)Z_t^x \Big) dt + dB_t^x, & t \in [0,T],\ x \in I \\ Z_t^x = \displaystyle\int_I w(x,y) X_t^y \lambda(dy), & t \in [0,T],\ x \in I, \\ X_0^x = \xi^x \end{cases} \tag{1}$$

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Theorem (A., Carmona, Laurière)
Let $a, b, c : I \to \mathbb{R}$ be $\mathcal{I}$-measurable and bounded. For each admissible strategy profile α (open-loop, decentralized, progressive, square-integrable):

→ *There exists a unique solution X to (1) (in $L^2_{\boxtimes}$-sense)*

→ *The aggregate Z is almost surely deterministic:*

$$\mathbb{P} \boxtimes \lambda\left(\|Z - f\|_T = 0\right) = 1, \quad f(\omega, x) := \widetilde{f}(x),\ \widetilde{f} \in L^2_\lambda(I;\mathcal{C}) \quad (\omega, x) \in \Omega \times I.$$

→ *There is a version of X that solves (1) for all $x \in I$ (in L^2-sense) and of Z that is deterministic for all $x \in I$.*

The Linear Graphon SDE System

Take-aways from the theorem:

→ We can replace Z_t^x with the **deterministic aggregate** $\int_I w(x,y)\mathbb{E}[X_t^y]\lambda(dy)$

→ or $\int_I w(x,y)\mathbb{E}[X_t^y]dy$ if $y \mapsto \mathbb{E}[X_t^y]$ is $\mathcal{B}(I)$-measurable (depends on assumptions on ξ, a, b, c)

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→ We can solve the Graphon SDE **"x-by-x"** (compared to an $L^2_\boxtimes$ sense)

The Stochastic Graphon Game

The Stochastic Graphon Game

$$J^x(\beta;\underline{\alpha}) := \mathbb{E}\Big[\int_0^T f^x\big(X_t^x, \beta_t, Z_t^x\big)dt + h^x\big(X_T^x, Z_T^x\big)\Big],$$

$$dX_t^x = \Big(a(x)X_t^x + b(x)\beta_t + c(x)Z_t^x\Big)dt + dB_t^x,\ X_0^x = \xi^x,$$

$$Z_t^x = \int_I w(x,u)\mathbb{E}[X_t^u]\lambda(du),\ u \in I,\ t \in [0,T]$$

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$\rightarrow$ Strategy profile $\underline{\alpha}$ appears *only in the aggregate*

The Stochastic Graphon Game

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\begin{aligned}
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→ Strategy profile $\underline{\alpha}$ appears *only in the aggregate*

→ Aggregate insensitive to changing one strategy (λ atomless)

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→ Strategy profile $\underline{\alpha}$ appears *only in the aggregate*

→ Aggregate insensitive to changing one strategy (λ atomless)

→ We write $J^x(\beta;\underline{\alpha})$ as $\mathcal{J}^x(\beta;Z^x)$

Definition (Nash equilibrium)

An admissible strategy profile $\underline{\hat{\alpha}}$ is a **graphon game Nash equilibrium** if

$$\mathcal{J}^x(\hat{\alpha}^x;Z^{\underline{\hat{\alpha}};x}) \le \mathcal{J}^x(\beta;Z^{\underline{\hat{\alpha}};x}),\quad \beta\in\mathcal{A}(x),\; x\in I$$

where $\mathcal{A}(x)$ is the set of decentralized, open-loop, progressive, square-integrable processes.

The Stochastic Graphon Game

Pontryagin Stochastic Maximum Principle

If $\underline{\hat{\alpha}}$ is a **graphon game Nash equilibrium** then

$$\hat{\alpha}_t^x \in \arg\inf_{u \in \mathbb{R}} H^x(t, \hat{X}_t^x, u, p_t^x), \quad \text{a.e. } t \in [0, T], \ \mathbb{P}\text{-a.s},$$

for each $x \in I$, with $(\hat{X}^x, p^x, q^x)$ solving the **Hamiltonian system**

$$\begin{cases} d\hat{X}_t^x = \partial_p H^x(t, \hat{X}_t^x, \hat{\alpha}_t^x, p_t^x)dt + dB_t^x, & \hat{X}_0^x = \xi^x, \\ dp_t^x = -\partial_\chi H^x(t, \hat{X}_t^x, \hat{\alpha}_t^x, p_t^x)dt + q_t^x dB_t^x, & p_T^x = \partial_\chi h^x(\hat{X}_T^x, \hat{Z}_T^x), \end{cases}$$

where $H^x : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the **Hamiltonian** of player x,

$$H^x(t, \chi, u, p) = f^x(\chi, u, \hat{Z}_t^x) + (a(x)\chi + b(x)u + c(x)\hat{Z}_t^x)p,$$

and $\hat{Z}_t^x$ is the aggregate of $\hat{X}_t^{\cdot}$: $\hat{Z}_t^x = \int_I w(x, y)\mathbb{E}[\hat{X}_t^y]\lambda(dy)$.

Sufficient condition when: $(\chi, u) \mapsto (f^x(\chi, u, z), h^x(\chi, z))$ is jointly convex for $z \in \mathbb{R}$.

The Stochastic Graphon Game

Linear-quadratic type assumptions:

→ f, h quadratic functions

→ a, b, c, f, h such that some Riccati equations are solvable

Then the Hamiltonian system (FBSDE) from the Pontryagin SMP

$$\begin{cases} d\hat{X}_t^x = \partial_p H^x(t, \hat{X}_t^x, \hat{\alpha}_t^x, p_t^x)dt + dB_t^x, \qquad \hat{X}_0^x = \xi^x, \\ dp_t^x = -\partial_\chi H^x(t, \hat{X}_t^x, \hat{\alpha}_t^x, p_t^x)dt + q_t^x dB_t^x, \quad p_T^x = \partial_\chi h^x(\hat{X}_T^x, \hat{Z}_T^x), \\ H^x(t, \chi, u, p) = f^x(\chi, u, \hat{Z}_t^x) + (a(x)\chi + b(x)u + c(x)\hat{Z}_t^x)p, \\ \hat{Z}_t^x = \int_I w(x, y)\mathbb{E}[\hat{X}_t^y]\lambda(dy) \end{cases}$$

has a **unique solution for arbitrary T and all $x \in I$ (in L^2-sense)**.

Proof idea:

1. Uniqueness in $L^2_\boxtimes$-sense by comparing two solutions
2. Existence in $L^2_\boxtimes$-sense for small T by contraction argument
3. Extend 2. to arbitrary T with the induction method for FBSDEs (Delarue '02)
4. Extract version solving the system for all $x \in I$

A Solvable Example

Example

$$\mathcal{J}^x(\alpha^x; Z^x) = \frac{1}{2}\mathbb{E}\Big[\int_0^{T=3} \big((\alpha_t^x)^2 + (X_t^x - Z_t^x)^2\big)\, dt + (X_T^x - Z_T^x)^2\Big]$$

$$dX_t^x = (-X_t^x + \alpha_t^x + Z_t^x)\, dt + dB_t^x, \quad X_0^x = \xi^x \sim \text{Normal}(8, 1/4),$$

$$Z_t^x = \int_I w(x,y)\mathbb{E}[X_t^y]\lambda(dy), \quad x \in I,\ t \in [0, T].$$

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f, h convex $\Rightarrow$ sufficient Pontryagin SMP, equilibrium characterized by the FBSDE

$$d\hat{X}_t^x = \left(-\hat{X}_t^x - p_t^x + \hat{Z}_t^x\right) dt + dB_t^x, \qquad \hat{X}_0^x = \xi^x$$

$$dp_t^x = \left(\hat{X}_t^x + p_t^x - \hat{Z}_t^x\right) dt + q_t^x\, dB_t^x, \qquad p_T^x = \hat{X}_T^x - \hat{Z}_T^x$$

$$\hat{Z}_t^x = \int_I w(x,y)\mathbb{E}[\hat{X}_t^y]\lambda(dy), \quad x \in I,\ t \in [0,T]$$

→ Solve the FBSDE explicitly up to a system of ODEs (some of them Riccati)

→ Size of ODE system determined by the rank of the graphon

Example

To solve the FBSDE, make the ansatz $p_t^x = \eta_t^x + \zeta_t^x \hat{X}_t^x$ where

$\rightarrow$ η^x and ζ^x are deterministic functions of time for all $x \in I$

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$$\begin{cases} \dfrac{d\eta_t^x}{dt} = (\eta_t^x)^2 + \eta_t^x + 1, & \eta_T^x = 1 \\ \dfrac{d\zeta_t^x}{dt} = (1+\eta_t^x)\zeta_t^x - (1+\eta_t^x)\hat{Z}_t^x, & \zeta_T^x = -\hat{Z}_T^x, \\ d\hat{X}_t^x = \left(-(1+\eta_t^x)\hat{X}_t^x - \zeta_t^x + \hat{Z}_t^x\right) dt + dB_t^x, & \hat{X}_0^x = \xi^x \end{cases}$$

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→ η^x independent of x (we drop the superscript)

→ Given η, $(\zeta, \hat{Z})$ forms a closed (infinite-dimensional) system

$$\begin{cases} \dfrac{d\zeta_t^x}{dt} = (1+\eta_t)\zeta_t^x - (1+\eta_t)\hat{Z}_t^x, & \zeta_T^x = -\hat{Z}_T^x, \\ \dfrac{d\hat{Z}_t^x}{dt} = -(1+\eta_t)\hat{Z}_t^x - [W\zeta_t^{\cdot}]^x + [W\hat{Z}_t^{\cdot}]^x, & \hat{Z}_0^x = [W\xi^{\cdot}]^x \end{cases}$$

Example

The graphon operator is Hilbert-Schmidt

- → $[W\zeta_t^{\cdot}]^x = \sum_{k=1}^{\infty} \lambda_k \phi_k(x) \langle \zeta, \phi_k \rangle_{\lambda_I}$
- → $\{\phi_k\}_{k=1}^{\infty}$ is an orthonormal basis in $L^2(I)$ of eigenfunctions of W
- → $\{\lambda_k\}_{k=1}^{\infty}$ are the corresponding eigenvalues

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Let $v_t^k := \langle \zeta_t, \phi_k \rangle_{\lambda_I}$, $z_t^k := \langle \hat{Z}_t, \phi_k \rangle_{\lambda_I}$, and $x^k := \langle \xi, \phi_k \rangle_{\lambda_I}$. Then

$$[W\hat{Z}_t^{\cdot}](x) = \sum_{k=1}^{\infty} \lambda_k z_t^k \phi_k(x), \qquad [W\zeta_t^{\cdot}](x) = \sum_{k=1}^{\infty} \lambda_k v_t^k \phi_k(x).$$

where for $k = 1, 2, \ldots$

$$\begin{cases} \dfrac{dv_t^k}{dt} = (1+\eta_t)v_t^k - (1+\eta_t)z_t^k, & v_T^k = -z_T^k, \\ \dfrac{dz_t^k}{dt} = \big(-1-\eta_t+\lambda_k\big)z_t^k + -\lambda_k v_t^k, & z_0^k = \lambda_k x^k. \\ x^k = [W\xi^{\cdot}]^x \overset{ELLN}{=} [W\mathbb{E}[\xi^{\cdot}]]^x = 8 \end{cases} \tag{2}$$

→ Size of FBODE system (2) is determined by the rank of W!
→ FBODE system (2) can be solved explicity with the ansatz $v_t^k = \pi_t^k z_t^k$

Example

Graphon	Form	Rank	Eigenvalue(s)	Eigenvector(s)
Constant	K	1	K	1
Power law	$(xy)^\gamma,\ \gamma \geq 0$	1	$(1-2\gamma)^{-1}$	$(1-2\gamma)^{-1/2}x^{-\gamma}$
Min-max	$(x \wedge y)(1 - x \vee y)$	∞	$(\pi k)^{-2},\ k \in \mathbb{N}$	$\sqrt{2}\sin(\pi k x),\ k \in \mathbb{N}$

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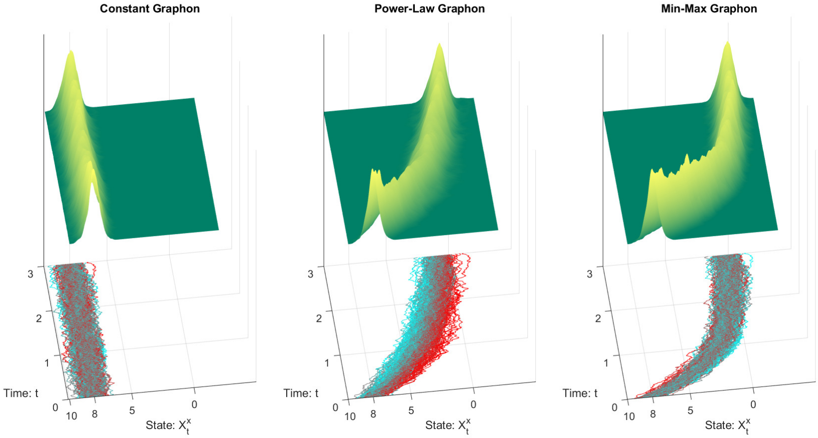

Connection with N-player games

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Two ways to construct finite graphs from graphons

$\rightarrow$ Sampling open/closed edges

$\rightarrow$ Weighing edges

We focus on connecting the **latter approach** to the graphon game.

Connection with N-player games

Two ways to construct finite graphs from graphons

→ Sampling open/closed edges

→ Weighing edges

We focus on connecting the **latter approach** to the graphon game.

→ I^∞ denote the countable product of I. A generic sequence $(x_k)_{k=1}^\infty$ in I^∞ will be denoted by x^∞.

→ $\mathcal{I}^\infty$ the countable product of $\mathcal{I}$

→ λ^∞ the countable product of λ

In **the iteratively completed infinite product space $(I^\infty, \bar{\mathcal{I}}^\infty, \bar{\lambda}^\infty)$** the processes $(B^x)_{x \in x^\infty}$ are mutually independent for $\bar{\lambda}^\infty$-a.e. $x^\infty \in I^\infty$ (Hammond, Sun '21).

Connection with N-player games

→ Let $(x_k)_{k=1}^{\infty} = x^{\infty} \in I^{\infty}$ be given

→ Consider the N-player game

$$J^{k,N}(\alpha^{k,N};\alpha^{-k,N}) := \mathbb{E}\Big[\int_0^T f^{x_k}(X_t^{k,N},\alpha_t^{k,N},Z_t^{k,N})dt + h^{x_k}(X_T^{k,N},Z_T^{k,N})\Big]$$

$$dX_t^{k,N} = \big(a(x_k)X_s^{k,N} + b(x_k)\alpha_t^{k,N} + c(x_k)Z_t^{k,N}\big)dt + dB_t^{x_k}, \quad X_0^{k,N} = \xi^{x_k},$$

$$Z_t^{k,N} := \frac{1}{N}\sum_{\ell=1}^{N} w(x_k,x_\ell)X_t^{\ell,N}, \quad k = 1,\dots,N, \; t \in [0,T].$$

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Equilibrium conditions by Pontryagin SMP: a fully coupled FBSDE system for

$$(\hat{X}^{k,N}, p^{k\ell,N}, q^{k\ell m,N})_{k,\ell,m=1}^{N}$$

Connection with N-player games

Propagation of Chaos

$$\Delta(x^\infty, N) :=$$
$$\max_{1\le k\le N} \left(\mathbb{E}\left[\sup_{t\in[0,T]} \left(|\hat{X}_t^{k,N} - \hat{X}_t^{x_k}|^2 + |p_t^{kk,N} - p_t^{x_k}|^2\right)\right] + \sup_{t\in[0,T]} \mathbb{E}\left[|\hat{Z}_t^{k,N} - \hat{Z}_t^{x_k}|^2\right]\right).$$

Theorem (A., Carmona, Laurière)

For $\bar{\lambda}^\infty$-a.e. $x^\infty \in I^\infty$: $\Delta(x^\infty, N) \xrightarrow[N\to+\infty]{} 0$

Connection with N-player games

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Theorem (A., Carmona, Laurière)

For $\bar{\lambda}^\infty$-a.e. $x^\infty \in I^\infty$: $\Delta(x^\infty, N) \xrightarrow[N\to+\infty]{} 0$

If furthermore $I \ni x \mapsto w(x,y) \in \mathbb{R}$ is 1/2-Hölder continuous, uniformly in $y \in I$, then for all $\varepsilon > 0$ there exists a $N_\varepsilon : I^\infty \to \mathbb{N}$ such that

$$\bar{\lambda}^\infty\left(\Delta(x^\infty, N) \leq \frac{(C+\varepsilon)^2 \log\log N}{N},\ N \geq N_\varepsilon(x^\infty)\right) = 1,$$

where C is a finite positive constant depending only on T and the graphon w.

$\to$ Similar result under other conditions, we can avoid the continuity assumption

Results on the connection with N-player games implied by the PoC result:

→ The graphon game Nash equilibrium strategy collection $(\hat{\alpha}^{x_k})_{k=1}^{N}$ **forms an ε_N-Nash equilibrium for the N-player game** between the players $(x_1, \ldots, x_N)$ when $N \geq \underline{N}(x^\infty)$, $\bar{\lambda}^\infty$-a.s. where $\varepsilon_N = O(N^{-1} \log\log N)$.

→ The N-player game Nash equilibrium **converges componentwise to the graphon game Nash equilibrium**; the rate of convergence is uniform and at most ε_N:

$$\max_{1 \leq k \leq N} \mathbb{E}\Big[\int_0^T |\hat{\alpha}_t^{k,N} - \hat{\alpha}_t^{x_k}|^2 dt\Big] \leq \varepsilon_N^2, \quad N \geq \underline{N},\ \bar{\lambda}^\infty\text{-a.e. } x^\infty \in I^\infty.$$

Finite State Stochastic Graphon Games

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Consider a game between càdlàg processes with a finite state space $E = \{1, \ldots, M\}$.

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→ Aggregates are deterministic **and continuous** by ELLN (for a carefully chosen class of controls).

SIR transition rate matrices

$$\begin{bmatrix} \cdots & \beta p_t(I) & 0 \\ 0 & \cdots & \gamma \\ 0 & 0 & \cdots \end{bmatrix} \text{ vs. } \begin{bmatrix} \cdots & \beta \int_I w(x,y) p_t^y(I) dy & 0 \\ 0 & \cdots & \gamma \\ 0 & 0 & \cdots \end{bmatrix}$$

Finite State Stochastic Graphon Games

Consider random processes with a finite state space $E = \{1, \dots, M\}$.

What changes?

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→ Pure-jump SDE describes state

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Controlled SIR transition rate matrices

$$\begin{bmatrix} \cdots & \beta\alpha_t \int_A a\rho_t(I, da) & 0 \\ 0 & \cdots & \gamma \\ 0 & 0 & \cdots \end{bmatrix} \text{ vs. } \begin{bmatrix} \cdots & \beta\alpha_t^x \int_I w(x,y) \left(\int_A a\rho_t^y(I, da) \right) dy & 0 \\ 0 & \cdots & \gamma \\ 0 & 0 & \cdots \end{bmatrix}$$

Finite State Stochastic Graphon Games

→ Pure-jump SDE representation with Poisson random measures $(N^x_\cdot)_{x \in I}$

→ Extended mean-field interaction to model epidemic disease spread

What we know

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→ Extended mean-field interaction to model epidemic disease spread

What we know

→ The Graphon SDE system is well-defined (for a carefully chosen class of controls)

Theorem (A., Carmona, Dayanıklı, Laurière)
Fix an admissible strategy profile $\underline{\alpha}$. If κ and K are bounded and Lipschitz, then there exists a unique solution X (in $L^2_\boxtimes$-sense), càdlàg and E-valued, to

$$X^x_t = \xi^x + \sum_{k=-n+1}^{n-1} k \int_{\mathbb{R}\times(0,t]} 1_{[0,\kappa^x_s(X^x_{s-},k,\alpha^x_s,Z^x_{s-})]}(y) N^x_k(dy \otimes ds),$$

$$Z^x_t = \int_I w(x,y) K(\alpha^y_t, X^y_{t-}) \lambda(dy),$$

the corresponding aggregate Z is $\mathbb{P} \boxtimes \lambda$-a.s. deterministic and continuous, and there is a version solving the system for all $x \in I$ in standard L^2-sense.

Finite State Stochastic Graphon Games

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→ There exists a solution to the analytic game (FBODE system)

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What is still to be done

→ Probabilistic formulation of the game equilibrium

→ Connection to N-player games

Thank you!