Mean-field type control of BSDEs: pedestrian movement to a target location

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1. Motivation

- A tool for centrally planned desicion-making for the movement of pedestrian groups who are forced to reach a target position, such as a security team.
- In a dense crowd, pedestrians react to crowd aggregates which, if all agents are similar, can be replaced by the law of a typical agent.
- Mean-field control is suitable for pedestrian crowd modeling when
  a) the central planner is rational and has the ability to anticipate the behaviour of other pedestrians, and
  b) aggegate effects are considered.
- This approach leads to the formulation of an optimization problem.

2. Model formulation

- The motion of our agents is described by the BSDE

$$\begin{cases} dY_t = b(t, Y_t, \mathbb{P}_{Y_t}, Z_t, u_t)dt + Z_t dB_t, \\ Y_T = y_T. \end{cases} \quad (1)$$

- The agent uses two controls:
  - $(u_t)_t$, used to heed preferences on energy use etc. Picked from $\mathscr{U}[0,T]$, a set of admissible controls, by an optimization procedure.
  - $(Z_t)_t$, used to make the path to y_T the best prediction based on available information. Given implicitly by the martingale representation theorem.
- The central planner solves a mean-field type control problem,

$$\min_{u\in\mathscr{U}[0,T]} \mathbb{E}\left[\int_0^T f(t, Y_t, \mathbb{P}_{Y_t}, u_t)dt + h(Y_0, \mathbb{P}_{Y_0})\right], \quad (2)$$

given (1).

3. Optimal control

- A spike pertubation technique leads to a Pontryagin type maximum principle [1].

Theorem

Suppose that $(\hat{Y}, \hat{Z}, \hat{u})$ is a solution to the control problem (1)-(2). Let H be the Hamiltonian

$$H(t,y,\mu,z,u,p) = b(t,y,\mu,z,u)p - f(t,y,\mu,u)$$

where $(p_t)_t$ solves the adjoint equation

$$\begin{cases} dp_t = -\Big\{\partial_y H(t, \hat{Y}_t, \mathbb{P}_{\hat{Y}_t}, \hat{Z}_t, \hat{u}_t, p_t) \\ \qquad - \mathbb{E}\big[\partial_\mu H(t, \hat{Y}_t, (\mathbb{P}_{\hat{Y}_t})^*, \hat{Z}_t, \hat{u}_t, p_t\big]\Big\}dt \\ \qquad - p_t \partial_z b(t, \hat{Y}_t, \mathbb{P}_{\hat{Y}_t}, \hat{Z}_t, \hat{u}_t)dB_t, \\ p_0 = \partial_y h(\hat{Y}_0, \mathbb{P}_{\hat{Y}_t}) + \mathbb{E}\big[\partial_\mu h(\hat{Y}_t, (\mathbb{P}_{\hat{Y}_t})^*\big]. \end{cases}$$

Then

$$\hat{u}_t = \underset{u\in U}{\text{argmax}}\ H(t, \hat{Y}_t, \mathbb{P}_{\hat{Y}_t}, \hat{Z}_t, u, p_t), \quad \text{(MP)}$$

for a.e. t, $\mathbb{P}$-a.s.

Theorem

Suppose that H is concave in (y,μ,z,u), h is convex in (y,μ) and $(\hat{u}_t)_t$ satisfies (MP) $\mathbb{P}$-a.s. for a.e. t. Then $(\hat{Y}, \hat{Z}, \hat{u})$ solves the control problem (1)-(2).

4. Numerical example

- The goal of the central planner will be to keep our agents close together and to control their initial position. The agents' velocity is their control and thier acceleration is subject to noise.

$$\begin{cases} \min_{u\in\mathscr{U}[0,T]} \mathbb{E}\left[\int_0^T \lambda_1 u_t^2 + \lambda_2 (Y_t - \mathbb{E}[Y_t])^2 dt + \lambda_3 (Y_0 - y_0)^2\right] \\ \text{s.t. } dY_t = (u_t + B_t)dt + Z_t dB_t, \\ \qquad Y_T = y_T \end{cases}$$

- In Figure 1 the pedestrian density is plotted for the two sets of parameter values, the left row is mean-seeking ($\lambda_2 > 0$) while the right row is neutral ($\lambda_2 = 0$).
- The target is located at $y_T = (2,2)$ and the desired initial position is $y_0 = (0.1, 0.1)$.

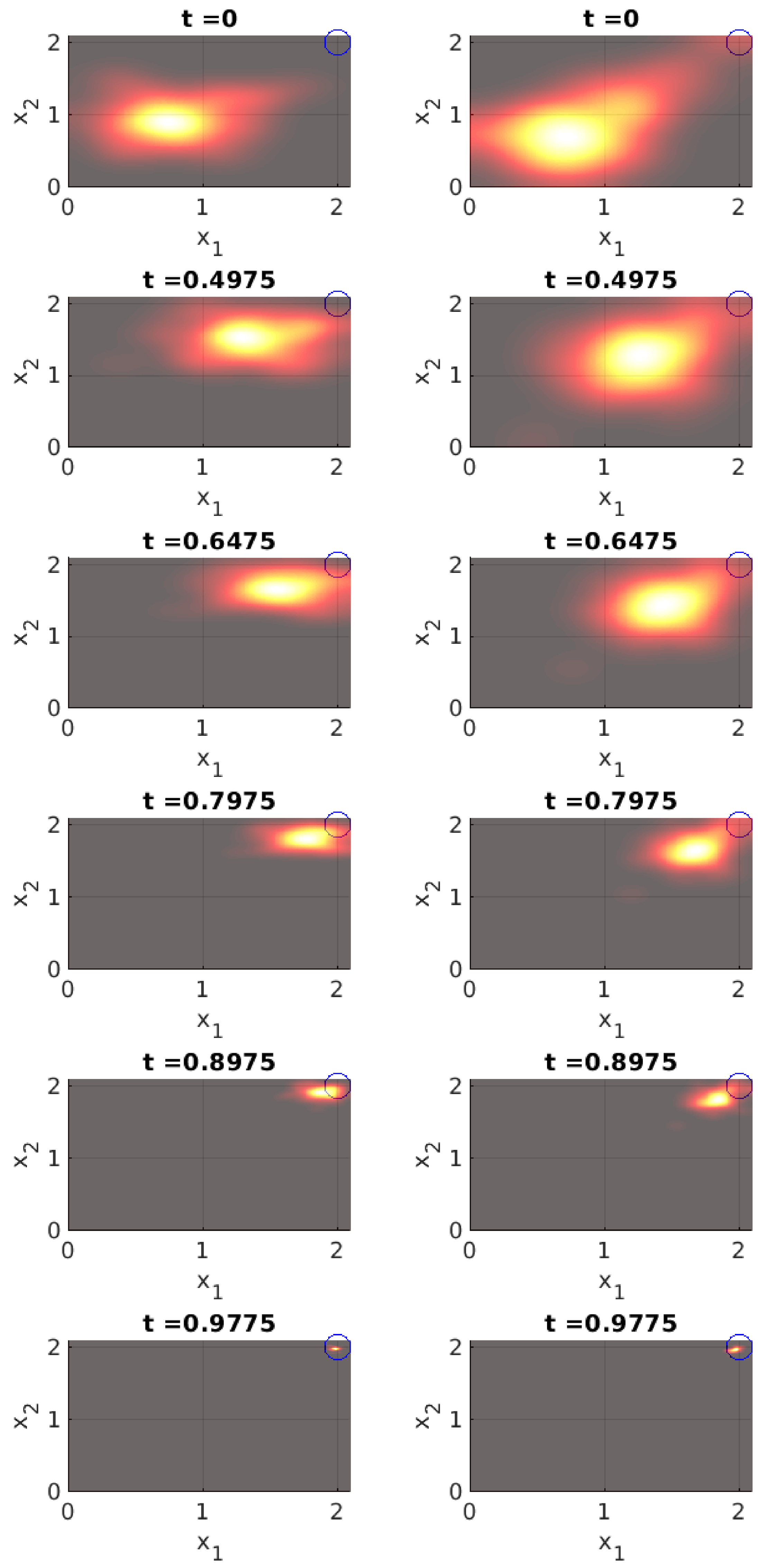

Figure 1. *Left row:* $(\lambda_1, \lambda_2, \lambda_3) = (50, 50, 10)$, *right row:* $(\lambda_1, \lambda_2, \lambda_3) = (50, 0, 10)$. Density estimate based on simulations of optimally controlled pedestrian paths. The pedestrians has a preferred starting region around $(0.1, 0.1)$ and will end up in $(2,2)$ (the blue circle). The pedestrian density is more concentrated for the left row, due to a mean-seeking preference.

Acknowledgements and references

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[1] Aurell, Alexander, and Boualem Djehiche. "Modeling tagged pedestrian motion: a mean-field type control approach." arXiv preprint arXiv:1801.08777 (2018)

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