On the mean-field type approach to crowd dynamics: the behavior of pedestrians near walls

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(Based on joint work with Boualem Djehiche (KTH))

Pedestrian crowds in confined domains

Example: Unidirectional pedestrian flow

Experimental results show that average pedestrian speed in a cross-section of a corridor can be higher in the center than near the walls[2], but also higher near the walls[3], depending on the circumstances (congestion, etc).

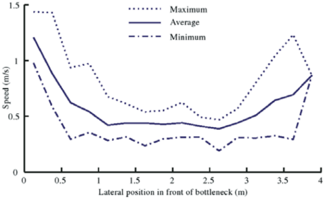

Fig. 5. Speeds as function of the lateral position in a cross-section upstream of the bottleneck during congestion.

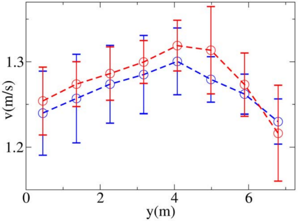

Figure 2. Velocity distributions as measured in the environment E_1 ($\bar{v}^+$ in red, $\bar{v}^-$ in blue). Error bars are obtained as standard deviations of values of $\bar{v}$ averaged over time windows of length 1200 s.
doi:10.1371/journal.pone.0050720.g002

[2] Winnie Daamen and Serge P Hoogendoorn. "Flow-density relations for pedestrian traffic". In: *Traffic and granular flow05*. Springer, 2007, pp. 315–322.

[3] Francesco Zanlungo, Tetsushi Ikeda, and Takayuki Kanda. "A microscopic social norm model to obtain realistic macroscopic velocity and density pedestrian distributions". In: *PloS one* 7.12 (2012), e50720.

Treatment of walls in pedestrian crowd models

Modeling approach	**Wall modeling**
Social force	Repulsive forces, disutility
Cellular automata (CA)	Forbidden cells
Continuum limit of CA	Neumann/no-flux boundary conditions
Hughes flow model	Neumann/no-flux boundary conditions, oblique reflection
Mean-field games/control/type games	Neumann/no-flux boundary conditions, disutility

Neumann/no-flux boundary conditions on the pedestrian density correspond to *reflection*.

Disutility

S Hoogendoorn and P Bovy. “Pedestrian route-choice and activity scheduling theory and models”. In: *Transportation Research Part B: Methodological* 38.2 (2004), pp. 169–190

C Dogbé. “Modeling crowd dynamics by the mean-field limit approach”. In: *Mathematical and Computer Modelling* 52.9-10 (2010), pp. 1506–1520

Neumann/No-flux

A Lachapelle and M-T Wolfram. “On a mean field game approach modeling congestion and aversion in pedestrian crowds”. In: *Transportation research part B: methodological* 45.10 (2011), pp. 1572–1589

M Burger et al. “On a mean field game optimal control approach modeling fast exit scenarios in human crowds”. In: *Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on*. IEEE. 2013, pp. 3128–3133

M Burger et al. “Mean field games with nonlinear mobilities in pedestrian dynamics”. In: *Discrete and Continuous Dynamical Systems-Series B* (2014)

M Cirant. “Multi-population mean field games systems with Neumann boundary conditions”. In: *Journal de Mathématiques Pures et Appliquées* 103.5 (2015), pp. 1294–1315

Y Achdou, M Bardi, and M Cirant. “Mean field games models of segregation”. In: *Mathematical Models and Methods in Applied Sciences* 27.01 (2017), pp. 75–113

In this talk we will introduce

sticky reflected SDEs of mean-field type with boundary diffusion

as an alternative approach to wall modeling in the mean-field approach to crowd dynamics.

Outline

1. Sticky reflected SDEs of mean-field type with boundary diffusion
2. Weak optimal control of sticky reflected SDEs of mean-field type with boundary diffusion
3. Particle picture
4. Example: Unidirectional pedestrian flow in a tight corridor

Sticky reflected SDEs of mean-field type with boundary diffusion

Consider the SDE system

$$\begin{cases} dX_t = \frac{1}{2} d\ell_t^0(X) + 1_{\{X_t>0\}} dB_t, \quad X_0 = x_0, \\ 1_{\{X_t=0\}} dt = \frac{1}{2\gamma} d\ell_t^0(X), \end{cases} \tag{1}$$

where

- $x_0 \in \mathbb{R}_+$,
- $\gamma \in (0, \infty)$ is a given constant,
- $\ell_0(X)$ is the local time of X at 0,
- B is a standard Brownian motion.

Engelberg and Peskir (2014)[2]:
System (1) has no strong solution but a unique weak solution, called a reflected Brownian motion X in $\mathbb{R}_+$ sticky at 0.

[2] Hans-Jürgen Engelbert and Goran Peskir. "Stochastic differential equations for sticky Brownian motion". In: *Stochastics An International Journal of Probability and Stochastic Processes* 86.6 (2014), pp. 993–1021.

Grothaus and Vosshall (2017)[2] extentend the result to a bounded domain $\mathcal{D} \subset \mathbb{R}^d$ with sticky C^2-smooth boundary $\partial\mathcal{D}$.

To write down the sticky reflected SDE with boundary diffusion system, let

- $n(x)$ be the outward normal of $\partial\mathcal{D}$ at x,
- $\pi(x) := E - n(x)(n(x))^*$, the orthogonal projection on the tangent space of $\partial\mathcal{D}$ at x,
- $\kappa(x) := (\pi(x)\nabla) \cdot n(x)$, the mean curvature of $\partial\mathcal{D}$ at x.

These quantities are uniformly bounded over $\partial\mathcal{D}$.

[2] Martin Grothaus, Robert Voßhall, et al. "Stochastic differential equations with sticky reflection and boundary diffusion". In: *Electronic Journal of Probability* 22 (2017).

Furthermore, let

- $\Omega := C([0, T]; \mathbb{R}^d)$ be path space,
- $\mathcal{F}$ the Borel σ-field over Ω,
- $X_t(\omega) = \omega(t)$ the coordinate process,
- $\mathbb{F}$ the $m \in \mathcal{P}(\Omega)$-completed filtration generated by X.

[2] Martin Grothaus, Robert Voßhall, et al. "Stochastic differential equations with sticky reflection and boundary diffusion". In: *Electronic Journal of Probability* 22 (2017).

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There exists a unique probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ under which

$$\begin{cases} dX_t = 1_{\mathcal{D}}(X_t)dB_t + 1_{\partial\mathcal{D}}(X_t)\left(dB_t^{\partial\mathcal{D}} - \frac{1}{2\gamma}n(X_t)dt\right), \\ dB_t^{\partial\mathcal{D}} = \pi(X_t)\circ dB_t = -\frac{1}{2}\kappa(X_t)n(X_t)dt + \pi(X_t)dB_t, \\ B \text{ standard Brownian motion in } \mathbb{R}^d,\ X_0 = x_0 \in \bar{\mathcal{D}},\ \gamma > 0, \end{cases}$$

and X is $C([0,T];\bar{\mathcal{D}})$-valued $\mathbb{P}$-a.s. (in particular, X is $\mathbb{P}$-a.s. uniformly bounded).[2]

[2] Martin Grothaus, Robert Voßhall, et al. "Stochastic differential equations with sticky reflection and boundary diffusion". In: *Electronic Journal of Probability* 22 (2017).

$$dX_t = (1_{\mathcal{D}}(X_t) + 1_{\partial\mathcal{D}}(X_t)\pi(X_t))\, dB_t - 1_{\partial\mathcal{D}}(X_t)\frac{1}{2}\left(\kappa(X_t) + \frac{1}{\gamma}\right) n(X_t)dt$$

The sticky reflected SDE with boundary diffusion is composed of

- interior diffusion $1_{\mathcal{D}}(X_t)dB_t$,
- boundary diffusion $1_{\partial\mathcal{D}}(X_t)dB_t^{\partial\mathcal{D}}$
- normal sticky reflection $-1_{\partial\mathcal{D}}(X_t)\frac{1}{2\gamma}n(X_t)dt$

From now on, we abbreviate

$$dX_t =: \sigma(X_t)dB_t + a(X_t)dt.$$

$$\sigma(X_t) := 1_{\mathcal{D}}(X_t) + 1_{\partial\mathcal{D}}(X_t)\pi(X_t),\ a(X_t) := -1_{\partial\mathcal{D}}(X_t)\frac{1}{2}\left(\kappa(X_t) + \frac{1}{\gamma}\right) n(X_t).$$

are bounded.

The stickiness level γ

γ represents the level of stickiness of $\partial\mathcal{D}$.

Let

- λ be the Lebesgue measure on $\mathbb{R}^d$,
- s be the surface measure on $\partial\mathcal{D}$,
- $\rho := 1_{\mathcal{D}}\alpha\lambda + 1_{\partial\mathcal{D}}\alpha' s, \quad \alpha, \alpha' \in \mathbb{R}$.

Choosing

$$\alpha = \bar{\alpha}/\lambda(\mathcal{D}),\ \alpha' = (1-\bar{\alpha})/s(\partial\mathcal{D}),\ \bar{\alpha} \in [0,1],$$

ρ becomes a probability measure on $\mathbb{R}^d$ with full support on $\bar{\mathcal{D}}$.

The measure ρ is in fact the invariant distribution of X_t whenever

$$\frac{1}{\gamma} = \frac{\bar{\alpha}}{(1-\bar{\alpha})}\frac{s(\partial\mathcal{D})}{\lambda(\mathcal{D})}.$$

$\bar{\alpha} \to 1$ as $\gamma \to 0$, and the invariant distribution ρ concentrates on $\mathcal{D}$

$\bar{\alpha} \to 0$ as $\gamma \to \infty$, and the invariant distribution ρ concentrates on $\partial\mathcal{D}$

Interaction and control is introduced via Girsanov transformation (Dominated case).

Let

- $|x|_t := \sup_{0\leq s\leq t} |x_s|$, $0 \leq t \leq T$,
- $U \subset \mathbb{R}^d$ be compact and $\mathcal{U} =: \{u : [0, T] \times \Omega \to U \mid u \ \mathbb{F}\text{-prog.meas.}\}$,
- $\mathbb{Q}(t) := \mathbb{Q} \circ X_t^{-1}$ denote the t-marginal distribution of X under $\mathbb{Q} \in \mathcal{P}(\Omega)$,
- $\beta : [0, T] \times \Omega \times \mathcal{P}(\mathbb{R}^d) \times U \to \mathbb{R}^d$ be a measurable function such that

(A) $(\beta(t, X, Q(t), u_t))_{t\leq T}$ is $\mathbb{F}$-prog.meas. for every $\mathbb{Q} \in \mathcal{P}(\Omega)$ and $u \in \mathcal{U}$.

(B) For every $t \in [0, T]$, $\omega \in \Omega$, $u \in U$, and $\mu \in \mathcal{P}(\mathbb{R}^d)$,

$$|\beta(t, x, \mu, u)| \leq C \left(1 + |x|_T + \int_{\mathbb{R}^d} |y|\mu(dy)\right),$$

(C) For every $t \in [0, T]$, $\omega \in \Omega$, $u \in U$, and $\mu, \mu' \in \mathcal{P}(\mathbb{R}^d)$,

$$\left|\beta(t, \omega, \mu, u) - \beta(t, \omega, \mu', u)\right| \leq C \cdot d_{TV}(\mu, \mu')$$

Given $\mathbb{Q} \in \mathcal{P}(\Omega)$ and $u \in \mathcal{U}$, let

$$L_t^{u,\mathbb{Q}} := \mathcal{E}_t \left(\int_0^{\cdot} \beta(s, X, \mathbb{Q}(s), u_s) dB_s \right).$$

Lemma 1

The positive measure $\mathbb{P}^{u,\mathbb{Q}}$ defined by $d\mathbb{P}^{u,\mathbb{Q}} = L_t^{u,\mathbb{Q}} d\mathbb{P}$ on $\mathcal{F}_t$, for all $t \in [0, T]$, is a probability measure on Ω. Moreover, under $\mathbb{P}^{u,\mathbb{Q}}$ the coordinate process satisfies

$$X_t = x_0 + \int_0^t \Big(\sigma(X_s)\beta(s, X, \mathbb{Q}(s), u_s) + a(X_s) \Big) ds + \int_0^t \sigma(X_s) dB_s^{u,\mathbb{Q}},$$

where $B_\cdot^{u,\mathbb{Q}}$ is a standard $\mathbb{P}^{u,\mathbb{Q}}$-Brownian motion.

Step 1. If φ is a process such that $\mathbb{P}^\varphi$, defined by $d\mathbb{P}^\varphi = L^\varphi_T d\mathbb{P}$ on $\mathcal{F}_T$ where $L^\varphi_t := \mathcal{E}_t(\int_0^\cdot \varphi_s dB_s)$, is a probability measure on Ω, the coordinate process under $\mathbb{P}^\varphi$ satisfies

$$dX_t = (\sigma(X_t)\varphi_t + a(X_t))\, dt + \sigma(X_t) dB^\varphi_t,$$

where B^φ is a $\mathbb{P}^\varphi$-Brownian motion. Smoothness of $\partial\mathcal{D}$ together with Burkholder-Davis-Gundy's inequality yields

$$\begin{aligned} E^\varphi[|X|^p_T] &\leq CE^\varphi\left[|x_0|^p + \int_0^T |\sigma(X_s)\varphi_s + a(X_s)|^p ds + \left|\int_0^\cdot \sigma(X_s) dB^\varphi_s\right|^p_T\right] \\ &\leq C\left(1 + \int_0^T E^\varphi[|\varphi_s|^p] ds\right), \end{aligned}$$

where E^φ denotes expectation taken under $\mathbb{P}^\varphi$.

Step 2. Consider the measure $\mathbb{P}_n^{u,\mathbb{Q}}$ given (on $\mathcal{F}_t$) by

$$d\mathbb{P}_n^{u,\mathbb{Q}} = \mathcal{E}_t\left(\int_0^\cdot \beta(s,X,\mathbb{Q}(s),u_s)1_{\{|X|_s\le n\}}dB_s\right)d\mathbb{P}.$$

Use TV-distance to show that $\mathbb{P}_n^{u,\mathbb{Q}} \in \mathcal{P}(\Omega)$. By Step 1, (B), and (C),

$$\begin{aligned}
E_n^{u,\mathbb{Q}}[|X|_T^p] &\le C\left(1+\int_0^T E_n^{u,\mathbb{Q}}\left[|\beta(s,X,\mathbb{Q}(s),u_s)|^p\right]ds\right)\\
&\le C\left(1+d_{TV}(\mathbb{Q}(s),\mathbb{P}(s))^p+\int_0^T E_n^{u,\mathbb{Q}}\left[|\beta(s,X,\mathbb{P}(s),u_s)|^p\right]ds\right)\\
&\le C\left(1+\int_0^T E_n^{u,\mathbb{Q}}\left[C\left(1+|X|_s^p+E^{\mathbb{P}}[|X|_s^p]\right)\right]ds\right)\\
&\le C\left(1+\int_0^T E_n^{u,\mathbb{Q}}[|X|_s^p]ds\right).
\end{aligned}$$

By Gronwall's inequality $E_n^{u,\mathbb{Q}}[|X|_T^p] \le C_p$, where C_p depends only on p, T, the Lipschitz and linear growth constant of β, and $|x_0|^p$.

Step 3. By the same lines as the proof of Proposition (A.1) in El-Karoui & Hamadène (2003)[2] (see also Benes (1971)[3]), the likelihood $L^{u,\mathbb{Q}}$ is a martingale for every $\mathbb{Q} \in \mathcal{P}(\Omega)$ and $u \in \mathcal{U}$, hence $\mathbb{P}^{u,\mathbb{Q}} \in \mathcal{P}(\Omega)$.

Step 4. By Girsanov's theorem the coordinate process under $\mathbb{P}^{u,\mathbb{Q}}$ satisfies

$$X_t = x_0 + \int_0^t \Big(\sigma(X_s)\beta(s, X, \mathbb{Q}(s), u_s) + a(X_s)\Big) ds + \int_0^t \sigma(X_s) dB_s^{\mathbb{Q}}.$$

■

[2] Nicole El-Karoui and Said Hamadène. "BSDEs and risk-sensitive control, zero-sum and nonzero-sum game problems of stochastic functional differential equations". In: *Stochastic Processes and their Applications* 107.1 (2003), pp. 145–169.

[3] VE Beneš. "Existence of optimal stochastic control laws". In: *SIAM Journal on Control* 9.3 (1971), pp. 446–472.

For a given $u \in \mathcal{U}$, consider the map

$$\Phi : \mathcal{P}(\Omega) \ni \mathbb{Q} \mapsto \mathbb{P}^{u,\mathbb{Q}} \in \mathcal{P}(\Omega).$$

Proposition 1

The map Φ is well-defined and admits a unique fixed point. Moreover, for every $p \geq 2$, the fixed point, denoted $\mathbb{P}^u$, belongs to $\mathcal{P}_p(\Omega)$, i.e.

$$E^u\left[|X|_T^p\right] \leq C_p < \infty,$$

where the constant C_p depends only on p, T, the Lipschitz and the linear-growth constant of β, and $|x_0|^p$.

Step 1. By Lemma 1, the map is well-defined.

Step 2. Given $\mathbb{Q}, \widetilde{\mathbb{Q}} \in \mathcal{P}(\Omega)$, by Csiszár-Kullback-Pinsker's inequality and the fact that $\int_0^{\cdot}(dB_s - \beta_s^{\mathbb{Q}} ds)$ is a martingale under $\Phi(\mathbb{Q})$,

$$
\begin{aligned}
&D_T^2(\Phi(\mathbb{Q}), \Phi(\widetilde{\mathbb{Q}})) \leq E^{\Phi(\mathbb{Q})}\left[\log(L_T^{\mathbb{Q}}/L_T^{\widetilde{\mathbb{Q}}})\right] \\
&= E^{\Phi(\mathbb{Q})}\left[\int_0^T (\beta_s^{\mathbb{Q}} - \beta_s^{\widetilde{\mathbb{Q}}}) dB_s - \frac{1}{2}\int_0^T (\beta_s^{\mathbb{Q}})^2 - (\beta_s^{\widetilde{\mathbb{Q}}})^2 ds\right] \\
&= E^{\Phi(\mathbb{Q})}\left[\int_0^T (\beta_s^{\mathbb{Q}} - \beta_s^{\widetilde{\mathbb{Q}}}) \beta_s^{\mathbb{Q}} - \frac{1}{2}(\beta_s^{\mathbb{Q}})^2 + \frac{1}{2}(\beta_s^{\widetilde{\mathbb{Q}}})^2 ds\right] \\
&= \frac{1}{2}\int_0^T \mathbb{E}^{\Phi(\mathbb{Q})}\left[(\beta_s^{\mathbb{Q}} - \beta_s^{\widetilde{\mathbb{Q}}})^2\right] ds \\
&\leq C \int_0^T d_{TV}^2(\mathbb{Q}(s), \widetilde{\mathbb{Q}}(s)) ds \leq C \int_0^T D_s^2(\mathbb{Q}, \widetilde{\mathbb{Q}}) ds.
\end{aligned}
$$

Step 3. Iterating the inequality, we obtain for every $N \in \mathbb{N}$,

$$D_T^2(\Phi^N(\mathbb{Q}), \Phi^N(\widetilde{\mathbb{Q}})) \le \frac{C^N T^N}{N!} D_T^2(\mathbb{Q}, \widetilde{\mathbb{Q}}),$$

where Φ^N denotes the N-fold composition of Φ. Hence Φ^N is a contraction for N large enough, thus admitting a unique fixed point.

Proof of Proposition 1

Step 3. Iterating the inequality, we obtain for every $N \in \mathbb{N}$,

$$D_T^2(\Phi^N(\mathbb{Q}), \Phi^N(\widetilde{\mathbb{Q}})) \leq \frac{C^N T^N}{N!} D_T^2(\mathbb{Q}, \widetilde{\mathbb{Q}}),$$

where Φ^N denotes the N-fold composition of Φ. Hence Φ^N is a contraction for N large enough, thus admitting a unique fixed point.

Step 4. Under $\mathbb{P}^u$, the fixed point of Φ given $u \in \mathcal{U}$, the coordinate process satisfies

$$dX_t = (\sigma(X_t)\beta(t, X, \mathbb{P}^u(t), u_t) + a(X_t))\, dt + \sigma(X_t) dB_t^u,$$

where B^u is a $\mathbb{P}^u$-Brownian motion. Following the calculations of Lemma 1, we get the estimate

$$\|\mathbb{P}^u\|_p^p = E^u[|X|_T^p] \leq C_p \left(1 + E^u\left[\int_0^T |X|_s^p ds\right]\right),$$

where C_p depends only on p, T, the Lipschitz and the linear growth constant of β, and $|x_0|^p$. Gronwall's inequality then yields $E^u\left[|X|_T^p\right] \leq C_p < \infty$.

■

Theorem 2

Under (A)-(C) there exists for each $u \in \mathcal{U}$ *a unique weak solution* $(\mathbb{P}^u)$ *to the sticky reflected SDE of mean-field type with boundary diffusion*

$$dX_t = \sigma(X_t)dB_t^u + \Big(a(X_t) + \sigma(X_t)\beta(t, X_t, \mathbb{P}^u(t), u_t)\Big)dt$$

Under $\mathbb{P}^u$ *the t-marginal distribution of* $X_\cdot$ *is* $\mathbb{P}^u(t)$ *for* $t \in [0, T]$ *and* $X_\cdot$ *is almost surely* $C([0, T]; \bar{\mathcal{D}})$*-valued. Furthermore,* $\mathbb{P}^u \in \mathcal{P}_p(\Omega)$*.*

Let

$$f : [0, T] \times \Omega \times \mathcal{P}(\mathbb{R}^d) \times U \to \mathbb{R},$$
$$g : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}.$$

Consider the following finite time-horizon problem:

$$\begin{cases} \min_{u \in \mathcal{U}} \; J(u) = E^u \left[\int_0^T f(t, X, \mathbb{P}^u(t), u_t) dt + g(X_T, \mathbb{P}^u(T)) \right] \\ \\ \\ \end{cases}$$

Let

$$f : [0, T] \times \Omega \times \mathcal{P}(\mathbb{R}^d) \times U \to \mathbb{R},$$
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Problem (2) is a weak form mean-field type control problem.
The probability space is controlled via the likelihood L^u.

Additional assumptions on β, f, and g:

(D) For $\phi \in \{\beta, f\}$,

$$\phi_t^u = \phi(t, X, E^u[r_\phi(X_t)], u_t) = \phi(t, X, E[L_t^u r_\phi(X_t)], u_t),$$

and $g_T^u = g(X_T, E[L_T^u r_g(X_T)])$, where $r_\beta, r_f, r_g : \mathbb{R}^d \to \mathbb{R}^d$.

(E) The functions $(t, x, y, u) \mapsto (f, \beta)(t, x, y, u)$ and $(x, y) \mapsto g(x, y)$ are twice continuously differentiable with respect to y. Moreover, β, f and g and all their derivatives up to second order with respect to y are continuous in (y, u), and bounded.

(D)-(E) can be relaxed, current form used for the sake of technical simplicity.

In view of (A)-(E) Pontryagin's type stochastic maximum principle is available[2].

Theorem 3

Assume that $(\hat{u}, L^{\hat{u}})$ is an optimal solution to the mean-field type control problem (2). Then for all $v \in U$ and a.e. $t \in [0, T]$ it holds $\mathbb{P}$-a.s. that

$$\mathcal{H}(L_t^{\hat{u}}, v, p_t, q_t) - \mathcal{H}(L_t^{\hat{u}}, \hat{u}_t, p_t, q_t) + \frac{1}{2}[\delta(L\beta)(t)]^T P_t[\delta(L\beta)(t)] \leq 0,$$

where

$$\mathcal{H}(L_t^u, u_t, p_t, q_t) := L_t^u \beta_t^u q_t - L_t^u f_t^u,$$

$$\delta(L\beta)(t) := L_t^{\hat{u}}(\beta(t, X, E[L_t^{\hat{u}} r_\beta(X_t)], v) - \beta_t^{\hat{u}}),$$

$$\begin{cases} dp_t = -\Big(q_t\beta_t^{\hat{u}} + E\left[q_t L_t^{\hat{u}} \nabla_y \beta_t^{\hat{u}}\right] r_\beta(X_t) - f_t^{\hat{u}} - E\left[L_t^{\hat{u}} \nabla_y f_t^{\hat{u}}\right] r_f(X_t)\Big) dt + q_t dB_t, \\ p_T = -g_T^{\hat{u}} - E\left[L_T^{\hat{u}} \nabla_y g_T^{\hat{u}}\right] r_g(X_T), \\ dP_t = -\Big(\left(\beta_t^{\hat{u}} + E[L_t^{\hat{u}} \nabla_y \beta_t^{\hat{u}}] r_\beta(X_t)\right)^2 P_t + 2\left(\hat{\beta}_t^{\hat{u}} + E[L_t^{\hat{u}} \nabla_y \beta_t^{\hat{u}}] r_\beta(X_t)\right) Q_t \\ \qquad\qquad + E[q_t \nabla_y \beta_t^{\hat{u}}] r_\beta(X_t) - E[\nabla_y f_t^{\hat{u}}] r_f(X_t)\Big) dt + Q_t dB_t, \\ P_T = 0, \end{cases}$$

[2] Rainer Buckdahn, Boualem Djehiche, and Juan Li. "A general stochastic maximum principle for SDEs of mean-field type". In: *Applied Mathematics & Optimization* 64.2 (2011), pp. 197–216.

Whenever U is convex, the optimality condition simplifies to

$$\mathcal{H}(L_t^{\hat{u}}, v, p_t, q_t) - \mathcal{H}(L_t^{\hat{u}}, \hat{u}_t, p_t, q_t) \leq 0, \quad \forall v \in U;\ \mathbb{P}\text{-a.s., a.e.-}t \in [0, T].$$

Assume that $\hat{u}$ is optimal. A matching argument yields

$$q_t = -\nabla_x \phi\,(X_t, t)\,\sigma(X_t),$$

where $\phi(X_T, T)$ is the terminal condition for p,

$$\phi(X_t, t) := g\left(X_t, E^{\hat{u}}[r_g(X_t)]\right) + E^{\hat{u}}\left[\nabla_y g\left(X_t, E^{\hat{u}}[r_g(X_t))]\right)\right] r_g(X_t),$$

and the optimality condition (variation of $\mathcal{H}$) relates $\hat{u}$ to q,

$$q_t \nabla_u \beta_t^{\hat{u}} = \nabla_u f_t^{\hat{u}}, \quad \mathbb{P}\text{-a.s., a.e.-}t \in [0, T].$$

Experimental results show that average pedestrian speed in a cross-section of a corridor can be higher in the center than near the walls[2], but also higher near the walls[3], depending on the circumstances (congestion, etc).

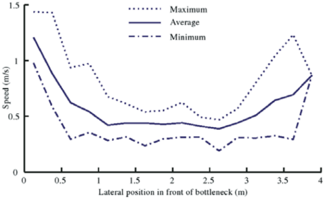


Fig. 5. Speeds as function of the lateral position in a cross-section upstream of the bottleneck during congestion.

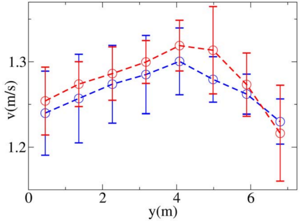


Figure 2. Velocity distributions as measured in the environment E_1 ($\bar{v}^+$ in red, $\bar{v}^-$ in blue). Error bars are obtained as standard deviations of values of $\bar{v}$ averaged over time windows of length 1200 s.
doi:10.1371/journal.pone.0050720.g002

[2] Winnie Daamen and Serge P Hoogendoorn. "Flow-density relations for pedestrian traffic". In: *Traffic and granular flow05*. Springer, 2007, pp. 315–322.

[3] Francesco Zanlungo, Tetsushi Ikeda, and Takayuki Kanda. "A microscopic social norm model to obtain realistic macroscopic velocity and density pedestrian distributions". In: *PloS one* 7.12 (2012), e50720.

Example: Unidirectional pedestrian flow

Let $\mathcal{D}$ be a long narrow corridor with exit x_T and entrance x_0 in opposite ends.

$$\begin{cases} \min_{u.\in\mathcal{U}} \frac{1}{2} E\left[\int_0^1 L_t^u f(t, X_\cdot, E[L_t^u r_f(X_t)], u_t)dt + L_T^u |X_T - x_T|^2\right], \\ \text{s.t. } dL_t^u = L_t^u u_t dB_t, \;\; L_0^u = 1. \end{cases}$$

f is a congestion-type running cost:

$$f(t, X_\cdot, E[L_t^u r_f(X_t)], u_t) = \mathcal{C}(X_t)\left\{1 + h\left(t, X_\cdot, E^u[r_f(X_t)]\right)\right\}|u_t|^2,$$

where

- $|u|^2$, $c_f > 0$, is the cost of moving in free space;
- $h|u|^2$ is the additional cost to move in congested areas;
- $\mathcal{C}(X_t) := \xi 1_\Gamma(X_t) + 1_\mathcal{D}(X_t)$, $\xi > 0$, monitors f on the boundary $\partial\mathcal{D}$.

Lower ξ yields lower overall cost of moving on $\partial\mathcal{D}$ and vice versa.

Example: Unidirectional pedestrian flow

Assuming U is convex, an optimal control satisfies

$$\hat{u}_t = \frac{\sigma(X_t)(X_t - x_T)}{\mathcal{C}(X_t)\left(1 + h(t, X_\cdot, E^{\hat{u}}[r_f(X_t)])\right)}, \quad \mathbb{P}\text{-a.s., a.e.-}t \in [0, T].$$

$\hat{u}$ implements the following strategy:

- Move towards the exit x_T, but scale the speed according to the local congestion.

Example: Unidirectional pedestrian flow

$$\hat{u}_t = \frac{\sigma(X_t)(X_t - x_T)}{\mathcal{C}(X_t)\left(1 + h(t, X_\cdot, E^{\hat{u}}[r_f(X_t)])\right)}.$$

We will compare two congestion costs

- friendly

$$h = h_1 := |X_2(t) - E^{\hat{u}}[X_2(t)]|$$

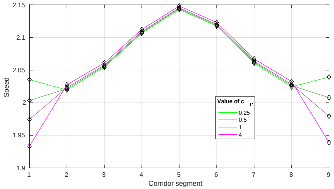
- averse

$$h = h_2 := \frac{1}{|X_2(t) - E^{\hat{u}}[X_2(t)]|}$$

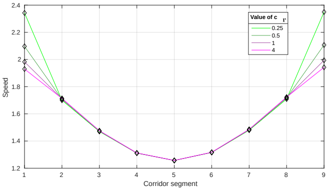
In both cases,

- $r_f((x_1, x_2)) = x_2$
- $X_2(t)$ is the y-component of X_t (perpendicular to the corridor walls).

Estimated cross-section mean speed profiles



(a) Congestion friendly ($h = h_1$).



(b) Congestion averse ($h = h_2$).

- Boundary movement speed is indeed monitored through ξ.

Consider $N \in \mathbb{N}$ (non-transformed, independent) sticky reflected SDEs with boundary diffusion

$$\begin{cases} dX_t^i = a(X_t^i)dt + \sigma(X_t^i)dB_t^i, \\ X_0^i = x_i, \quad i = 1, \dots, N. \end{cases} \tag{3}$$

Grothaus and Vosshall[2] (2017):

> There exists a unique probability measure $\mathbb{P}^N$ on $(\mathbf{\Omega}, \mathscr{F})$, where $\mathbf{\Omega} := C([0,T]; \mathbb{R}^{Nd})$ and $\mathscr{F}$ is the corresponding filtration. Under $\mathbb{P}^N$, $(X^1, \dots, X^N)$ satisfies (3) and is $C([0,T]; \bar{\mathcal{D}}^N)$-valued $\mathbb{P}^N$-a.s.

[2] Martin Grothaus, Robert Voßhall, et al. "Stochastic differential equations with sticky reflection and boundary diffusion". In: *Electronic Journal of Probability* 22 (2017).

Weak interaction and control can be introduced in the particle system[2]

Given $\mathbf{u} := (u^1, \dots, u^N) \in \mathcal{U}^N$, let $\mu^N(t) := \frac{1}{N} \sum_{i=1} \delta_{X_t^i}$ and

$$dL^{\mathbf{u}}_{i,t} = L^{\mathbf{u}}_{i,t} \beta(t, X^i, \mu^N(t), u^i_t) dB^i_t, \quad L^{\mathbf{u}}_{i,0} = 1, \quad i = 1, \dots, N.$$

$$L^{N,\mathbf{u}}_t := \prod_{i=1}^{N} L^{\mathbf{u}}_{i,t}.$$

$L^{N,\mathbf{u}}_t$ defines a Girsanov transformation of $\mathbb{P}^N$ to $\mathbb{P}^{N,\mathbf{u}}$.

Under $\mathbb{P}^{N,\mathbf{u}}$ the coordinate process is $C([0,T];\bar{\mathcal{D}})$-valued a.s. and satisfies

$$\begin{cases} dX^i_t = (\sigma(X^i_t)\beta(t, X^i_t, \mu^N(t), u^i_t) + a(X^i_t))dt + \sigma(X^i_t) dB^{i,\mathbf{u}}_t, \\ X^i_0 = x^i_0, \quad i = 1, \dots, N, \end{cases}$$

where $B^{i,\mathbf{u}}$ is a $\mathbb{P}^{N,\mathbf{u}}$-Brownian motion. Also, $\mathbb{P}^{N,\mathbf{u}} \in \mathcal{P}_p((C([0,T];\bar{\mathcal{D}})^N)$.

[2] Martin Grothaus, Robert Voßhall, et al. "Stochastic differential equations with sticky reflection and boundary diffusion". In: *Electronic Journal of Probability* 22 (2017).

Particle picture: The corresponding microscopic model

Social cost for the particle system:

$$J_N(\mathbf{u}) := \frac{1}{N}\sum_{i=1}^{N} E^{N,\mathbf{u}}\left[\int_0^T f(t, X^i, \mu^N(t), u_t^i)dt + g(X_T^i, \mu^N(T))\right]$$

Minimization of $J_N(\mathbf{u})$ is a cooperative scenario.

Mean-field type optimal control is $\epsilon(N)$-optimal for the collaborative social cost minimization, where $\epsilon(N) \to 0$ as $N \to \infty$. Based on results concerning convergence properties of relaxed controls.

Main references: El Karoui, Huu Nguyen and Jean-Blanc (1988)[2] (controlled standard SDEs), Ölschläger (1984)[3] (mean-field SDEs without control), Lacker (2017)[4] (controlled mean-field SDEs).

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[2] Nicole El Karoui, Du Huù Nguyen, and Monique Jeanblanc-Picqué. "Existence of an optimal Markovian filter for the control under partial observations". In: *SIAM journal on control and optimization* 26.5 (1988), pp. 1025–1061.

[3] Karl Oelschlager et al. "A martingale approach to the law of large numbers for weakly interacting stochastic processes". In: *The Annals of Probability* 12.2 (1984), pp. 458–479.

[4] Daniel Lacker. "Limit theory for controlled McKean–Vlasov dynamics". In: *SIAM Journal on Control and Optimization* 55.3 (2017), pp. 1641–1672.

Conclusions

- Mean-field approach to crowd dynamics
  - congestion, crowd aversion, etc.
  - decision-based modeling with anticipating agents
  - correspondence between micro- and macroscopic picture

- Sticky reflected SDEs of mean-field type with boundary diffusion
  - as an alternative to reflective boundary conditions in confined domains
  - pedestrians no longer "bounce" at the boundary
  - pedestrians may interact and take actions while spending time at the boundary
  - preserves a micro-macro correspondence for crowds in confined domains

Thank you!

Assume that $(\hat{u}, \hat{L})$ is an optimal solution for the mean-field type control problem. Recall the first order adjoint equation,

$$\begin{cases} dp_t = -\Big(q_t \beta_t^{\hat{u}} + E\left[q_t L_t^{\hat{u}} \nabla_y \beta_t^{\hat{u}} \right] r\beta(X_t) \\ \qquad\qquad - f_t^{\hat{u}} - E\left[L_t^{\hat{u}} \nabla_y f_t^{\hat{u}} \right] r_f(X_t) \Big) dt + q_t dB_t, \\ p_T = -g_T^{\hat{u}} - E\left[L_T^{\hat{u}} \nabla_y g_T^{\hat{u}} \right] r_g(X_T). \end{cases} \tag{4}$$

Rewriting $E[L_t^{\hat{u}} Y_t] = E^{\hat{u}}[Y_t]$ and changing measure to $\mathbb{P}^{\hat{u}}$,

$$\begin{cases} dp_t = -\Big(E^{\hat{u}}\left[q_t \nabla_y \beta_t^{\hat{u}} \right] r\beta(X_t) - f_t^{\hat{u}} - E^{\hat{u}}\left[\nabla_y f_t^{\hat{u}} \right] r_f(X_t) \Big) dt + q_t dB_t^{\hat{u}}, \\ p_T = -g_T^{\hat{u}} - E^{\hat{u}}\left[\nabla_y g_T^{\hat{u}} \right] r_g(X_T). \end{cases}$$

Examples: Convex and compact U

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Whenever U is convex, the optimality condition simplifies to

$$\mathcal{H}(\hat{L}_t, v, p_t, q_t) - \mathcal{H}(\hat{L}_t, \hat{u}_t, p_t, q_t) \leq 0, \quad \forall v \in U;\ \mathbb{P}\text{-a.s.},\ \text{a.e.-}t \in [0, T].$$

Example: Convex and compact U

p part of the solution to a BSDE so it is the conditional expectation

$$p_t = -E^{\hat{u}}\left[\phi\left(X_T, T\right) \mid \mathcal{F}_t\right] + E^{\hat{u}}\left[\int_t^T (\dots)ds \mid \mathcal{F}_t\right], \tag{5}$$

where as before

$$\phi(X_t, t) := g\left(X_t, E^{\hat{u}}[r_g(X_t)]\right) + E^{\hat{u}}\left[\nabla_y g\left(X_t, E^{\hat{u}}[r_g(X_t))]\right)\right] r_g(X_t).$$

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By Dynkin's formula,

$$E^{\hat{u}}[\phi(X_T, T) \mid \mathcal{F}_t] = \phi\left(X_t, t\right) + \int_t^T E^{\hat{u}}[(\dots)(s) \mid \mathcal{F}_t]ds.$$

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Itô-differentiating p from (5) and matching the diffusion coefficients yeilds

$$q_t = -\nabla_x \phi\left(X_t, t\right) \sigma(X_t).$$

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Itô-differentiating p from (5) and matching the diffusion coefficients yeilds

$$q_t = -\nabla_x \phi\left(X_t, t\right) \sigma(X_t).$$

The optimality condition (variation of $\mathcal{H}$) relates $\hat{u}$ to q,

$$q_t \nabla_u \beta_t^{\hat{u}} = \nabla_u f_t^{\hat{u}}, \quad \mathbb{P}\text{-a.s., a.e.-}t \in [0, T].$$

Example: Mean-field LQ (convex and compact U)

Consider on some admissible domain $\mathcal{D} \subset \mathbb{R}^d$ the mean-field LQ problem of minimizing final variance

$$\begin{cases} \min_{u \in \mathcal{U}} \dfrac{1}{2} E\left[\int_0^T L_t^u |u_t|^2 dt + L_T^u \left|X_T - E[L_T^u X_T]\right|^2\right], \\ \text{s.t. } dL_t^u = L_t^u u_t dB_t, \quad L_0^u = 1, \end{cases}$$

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The optimality condition says that $\hat{u}_t = q_t^*$ holds for an optimal control.

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$$\hat{u}_t = -(X_t - E^{\hat{u}}[X_t])^* \sigma(X_t), \ \mathbb{P}\text{-a.s. for almost every } t \in [0, T].$$

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$\hat{u}$ takes $\mathbb{P}$ to $\mathbb{P}^{\hat{u}}$ under which the coordinate process solves the non-linear SDE

$$dX_t = \left(a(X_t) - \sigma(X_t)(X_t - E^{\hat{u}}[X_t])\right) dt + \sigma(X_t) dB_t^{\hat{u}}.$$

Total variation distance on $\mathcal{P}(\Omega)$

For $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, the total variation distance is defined by the formula

$$d(\mu, \nu) = 2 \sup_{B \in \mathcal{B}(\mathbb{R}^d)} |\mu(B) - \nu(B)|. \tag{6}$$

Define on $\mathcal{F}$ the total variation metric

$$d(P, Q) := 2 \sup_{A \in \mathcal{F}} |P(A) - Q(A)|. \tag{7}$$

On the filtration $\mathbb{F}$,

$$D_t(Q, \widetilde{Q}) := 2 \sup_{A \in \mathcal{F}_t} |Q(A) - \widetilde{Q}(A)|, \quad 0 \leq t \leq T. \tag{8}$$

It satisfies

$$D_s(Q, \widetilde{Q}) \leq D_t(Q, \widetilde{Q}), \quad 0 \leq s \leq t. \tag{9}$$

For $Q, \widetilde{Q} \in \mathcal{P}(\Omega)$ with time marginals $Q_t := Q \circ x_t^{-1}$ and $\widetilde{Q}_t := \widetilde{Q} \circ x_t^{-1}$, then

$$d(Q_t, \widetilde{Q}_t) \leq D_t(Q, \widetilde{Q}), \quad 0 \leq t \leq T. \tag{10}$$

Endowed with the total variation metric D_T, $\mathcal{P}(\Omega)$ is a complete metric space. Moreover, D_T carries out the usual topology of weak convergence.