Sticky boundaries and boundary diffusion in the mean-field approach to pedestrian crowds

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1. Motivation

- In macroscopic models for pedestrian dynamics, interaction with walls is often modeled with Neumann-type boundary conditions on the crowd density. A drawback is that Neumann conditions on the density corresponds to a reflecting state process in the particle interpretation.
- Sticky boundaries and boundary diffusion has a 'smoothing' effect on pedestrian motion in the sense that when in use, pedestrian paths are on the boundary semi-martingales with first-variation part absolutely continuous with respect to dt.
- The state equation is an SDE with only a weak solution. This leads to the formulation of a weak control problem.

2. Model formulation

- Sticky reflected Brownian motion with boundary diffusion on bounded $\mathcal{D} \subset \mathbb{R}^D$

$$dX_t = \alpha(t,X_t)dt + \sigma(t,X_t)dB_t, \qquad (1)$$

where $\partial\mathcal{D}$ is C^2-smooth and

$$a(t,x) := -1_\Gamma(x)\frac{1}{2}\left(\frac{1}{\gamma}+\kappa(x)\right)n(x),$$
$$\sigma(t,x) := 1_{\mathcal{D}}(x) + 1_\gamma(x)\pi(x).$$

has unique weak solution $\mathbb{P}$ on path space $\Omega := C([0,T];\mathbb{R}^d)$ and $X. \in C([0,T];\bar{\mathcal{D}})$ $\mathbb{P}$-a.s. [1].

- Girsanov transformation used to introduce interaction and control.

$$\frac{d\mathbb{P}^u}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = L^u_t := \mathcal{E}_t\left(\int_0^\cdot \beta(t,X.,\mathbb{P}^u(t),u_t)dB_t\right),$$

Under $\mathbb{P}^u$, the coordinate process solves

$$dX_t = \Big(\sigma(t,X_t)\beta(t,X.,\mathbb{P}^u(t),u_t) + a(t,X_t)\Big)dt + \sigma(t,X_t)dB^u_t. \qquad (2)$$

- The goal is to minimize the cost $J(u) =$

$$E^u\Big[\int_0^T f(t,X.,\mathbb{P}^u(t),u_t)dt + g(X_T,\mathbb{P}^u(T))\Big]$$

over $u \in \mathcal{U}$ adapted U-valued processes, where $U \subset \mathbb{R}^d$ is compact.

3. Properties of (2)

- We assume that

$$|\beta(t,\omega,\mu,u)| \le C\left(1+|\omega|_T + \int_{\mathbb{R}^d}|y|\mu(dy)\right)$$
$$|\beta(t,\omega,\mu,u) - \beta(t,\omega,\mu',u)| \le C d_{TV}(\mu,\mu')$$

- Then (2) has a unique weak solution $\mathbb{P}^u \in (\mathcal{P}(\Omega), TV)$. Proof based on fixed-point technique that makes use of the Csiszár-Kullback-Pinsker inequality.
- Furthermore, $\mathbb{P}^u \in \mathcal{P}_p(\Omega)$ and the coordinate process is $C([0,T];\bar{\mathcal{D}})$-valued $\mathbb{P}^u$-a.s.

4. Control of (2)

- Minimizing J under the constraint that the coordinate process satisfies (2) is a weak mean-field type optimal control problem.
- Assume linear law-dependence. Then integration by parts yields $J(u) =$

$$E\Big[\int_0^T L^u_t f(t,X.,E[L^u_t r_f(X_t)],u_t)dt + L^u_T g(X_T, E[L^u_T r_g(X_T)])\Big] \qquad (3)$$

- Minimizing (3) under likelihood dynamics

$$dL^u_t = L^u_t\beta(t,X.,E[L^u_t r_\beta(X_t),u_t)dB_t$$

is a strong mean-field type control problem. There are available tools, like Pontryagin's type stochastic maximum principle.

- The likelihood has controlled diffusion, leading to second order adjoint equation. These are avoided if U is convex.

5. Microscopic interpretation of (2)-(3)

- The microscopic interpretation is valuable from the applied point of view since it allows us to study the crowd density to draw conclusions about individual behavior, and vice versa.
- System of $N \in \mathbb{N}$ sticky reflected Brownian motions with boundary diffusion,

$$\begin{cases} dX^i_t = a(t,X^i_t)dt + \sigma(t,X^i_t)dB^i_t, \\ X^i_0 = x_i, \quad i = 1,\dots,N, \end{cases}$$

- Given $\mathbf{u} = (u_1,\dots,u_N)$, let

$$dL^{\mathbf{u}}_{i,t} = L^{\mathbf{u}}_{i,t}\beta(t,X^i_\cdot,\mu^N_t,u^i_t)dB^i_t,$$

where $\mu^N_t := \frac{1}{N}\sum_{i=1}^N \delta_{X^i_t} \in \mathcal{P}(\mathbb{R}^d)$. $\prod_{i=1}^N L^{\mathbf{u}}_{i,t}$ takes $\mathbb{P}^N$ to $\mathbb{P}^{N,\mathbf{u}} \in \mathcal{P}_p(\Omega^N)$.

- Minimization of the social cost $J_N(\mathbf{u}) =$

$$\frac{1}{N}\sum_{i=1}^N E^{N,\mathbf{u}}\Big[\int_0^T f(t,X^i_\cdot,\mu^N_t,u^i_t)dt + g\left(X^i_T,\mu^N_T\right)\Big]$$

subject to the coordinate process solving

$$\begin{cases} dX^i_t = \Big(\sigma(t,X^i_t)\beta(t,X^i_\cdot,\mu^N_t,u^i_t) + a(t,X^i_t)\Big)dt + \sigma(t,X^i_t)dB^{i,\mathbf{u}}_t, \\ X^i_0 = x^i, \quad i = 1,\dots,N, \end{cases}$$

- Assume that the initial conditions are exchangeable, and

$$|f(t,\omega,\mu,u)| \le C\Big(1+|\omega|^p_T + \int_{\mathbb{R}^d}|y|^p\mu(dy) + |u|^p\Big),$$
$$|g(\omega,\mu)| \le C\Big(1+|\omega|^p_T + \int_{\mathbb{R}^d}|y|^p\mu(dy)\Big)$$

- We have the following approximation result:

> Let for each $N \in \mathbb{N}$ $\hat{\mathbf{u}}^N = (\hat{u},\dots,\hat{u}) \in \mathcal{U}^N$ where $\hat{u}$ is a minimizer of (3). Then $\lim_{N\to\infty} J_N(\mathbf{u}^N) =$
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> $$\mathbb{E}^{\hat{u}}\Big[\int_0^T f\left(t,X.,\mathbb{P}^{\hat{u}}(t),\hat{u}_t\right)dt + g\left(X_T,\mathbb{P}^{\hat{u}}(T)\right)\Big],$$
>
> where $E^{\hat{u}}$ is expectation taken under $\mathbb{P}^{\hat{u}}$. Therefore, $\hat{\mathbf{u}}^N$ is ϵ_N-optimal, where $\epsilon_N \to 0$ as $N \to \infty$, for the social cost optimization: For any $\mathbf{u}^N \in \mathcal{U}^N$,
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> $$J_N(\hat{\mathbf{u}}^N) - J_N(\mathbf{u}^N) \le \epsilon_N$$

6. Current research

- Generalization of approximation result that will give more detailed picture of the correspondence between the weak particle system and the weak MF system.
- Examples: The LQ-problem can be treated analytically.
- Application: Simulation study of pedestrian movement and speed profiles in confined domains.

Acknowledgements and references

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[1] Grothaus, Martin, and Robert Vosshall. "Stochastic differential equations with sticky reflection and boundary diffusion." Electronic Journal of Probability 22 (2017).